FUSION MATHEMATICS

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FUSION MATHEMATICS

A correlation and unification of Intermediate Algebra and Plane Trigonometry

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NEW YORK

SILVER, BURDETT AND COMPANY

NEW YORK · NEWARK · BOSTON
CHICAGO · SAN FRANCISCO

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at the same price.

Printed in the United States of America

PREFACE

The achievement of the educational objectives that should be realized by the study of mathematics necessitates a textbook that: (1) is written to the student; (2) consciously aims to develop in the student those qualities and attitudes that will better fit him for life in the modern world; (3) embodies a clear understanding of the varying needs, interests, and mental capacities of the student. With a clear understanding of these considerations and their educational implications, this book was prepared.

This book is written to the student. It is self-teaching. Each new concept is developed by statements and questions to be completed by the student. This leads him through his own activity to the discovery of new meanings. Afterwards he is presented with a simple, clear, and accurate statement of the results of his investigation and thinking. Finally, illustrative examples fully worked out clear up any remaining difficulty.

The motivation that introduces each new idea is planned to arouse the interest of the student and to show him his need for further study. Furthermore, each topic, as far as possible, is introduced when the work of the preceding topic makes it seem worth while to the student. For example, the binomial theorem is developed after he has raised several binomials to powers and has felt the need for a shorter and a general method.

This book aims to develop in the student those qualities and abilities considered most desirable as a preparation for life. It is built around, and places the major emphasis on, those basic ideas of mathematics which function most in life, such as the formula, graph, equation, and problem. Dependence, variation, and relationships are the unifying principles used to clarify and to show the unity of the subject matter. Note particularly the many sections called "A Study in Changes." These develop the function idea and at the same time show the relation of mathematics to life.

The book aims to develop intellectuality, the ability to read with precision and understanding, and habits of reflective thinking. To this end, after the careful development of a new idea, there will be found, under the heading "Something to think about," questions which test the student's understanding and stimulate him to do original thinking.

To make provision for the practice necessary to acquire needed skills and techniques, an abundance and variety of graded exercise material has been included. This work is a means to an end; namely, the development of the ability to solve *problems* involving original thinking. Persistent attention is paid to: (1) neat and orderly arrangement of computational work, partly because it is necessary for the avoidance of errors and is in itself a manifestation of clear thinking; and (2) accuracy of work as it fosters and develops self-confidence.

Great emphasis is placed on checking, and whenever feasible, more than one way of checking is given. Contact is made with the life needs of the student, as well as with his interests, in the many applications to business, science, engineering, and other fields of human activity.

The needs, interests, and varying mental capacities of the students are carefully provided for. This book is the work of authors who have had long experience as classroom teachers, and who now have teaching contact with students and know their needs, interests, and capacities. In addition, as supervisors of large departments of mathematics in schools totaling more than twenty thousand students, they have had ample opportunity to test the material and methods both with students and with teachers before incorporating them in this text.

The need for adapting the course to individual differences, and the time allotted to the work is provided for by a careful grading of the exercises into three groups, A, B, C. The A group contains enough material for a class of average ability or if the time allotted is restricted. The B group gives additional material for a somewhat more mature class or one that has more time to devote to the subject. The C group presents exercises that will challenge the determination and the mathematical ability of the student.

Special Features. In the treatment of logarithms, a new method of finding the characteristic from the number and of locating the decimal point from the characteristic, originated and experimented with by the authors, substitutes one simple procedure for the four rules so confusing to students.

In the treatment of graphs, several new developments are introduced, among them a method of finding a formula directly from a graph. Note the section called "From Pictures to Formulas."

Anticipating results is a very important function in everyday life. To prepare students to adequately meet this phase of life, this book stresses the value inherent in anticipating answers to problems. At the same time, this work serves as an additional check on the work as far as gross errors are concerned.

The Fusion Idea. In this book algebra and trigonometry are not treated as two separate subjects, but are so fused that the student is unaware that he is studying what heretofore have been two distinct courses. Such a fused course has several advantages over separate courses. There is a greater time exposure to the subjects, due to the fact that each can be used in the study of the other. There is also a considerable gain in time, inasmuch as many of the topics, such as logarithms and radicals, occur in both subjects and in a fusion course need be taught only once. Furthermore, the opportunity to make immediate use of these topics lends interest by making the reason for studying them more apparent to the student.

Much aid has been obtained from a careful study of the reports of investigations and recommendations of important educational committees, such as those of the National Committee on Mathematical Requirements. Such requirements as those of the College Entrance Examination Board, the New York State Regents, and modern courses of mathematics have been fully met.

THE AUTHORS.

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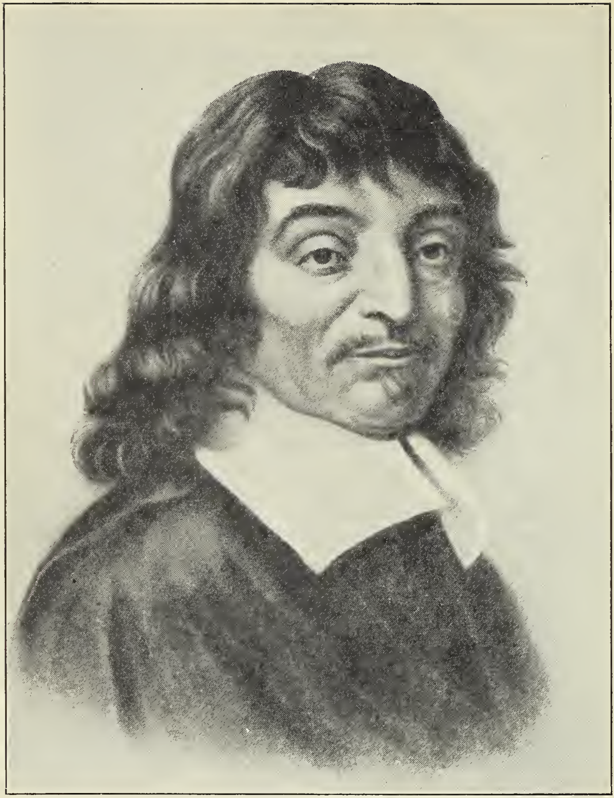


René Descartes (1596–1650)

From the portrait by Frans Hals, now in the Louvre, Paris. In addition to being a most distinguished philosopher, sometimes called the "father of modern philosophy," Descartes was a mathematician of note. To him we owe the system of using the letters x, y, and z to represent unknown quantities in algebra. He was the first to use Arabic numerals as exponents and introduced our present system of plotting points in work with graphs.

CHAPTER I. NUMERICAL TRIGONOMETRY

The advance and the perfecting of mathematics are closely joined to the prosperity of a nation. — Napoleon.

Indirect Measurement

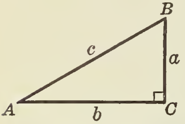
Do you know how to find the width of a river without swimming it? Do you remember how to find the distance through the base of a mountain without tunneling through it? Do you know how to find the height of your neighbor's flagpole without climbing it? Have you any idea of how our government determined that the area of the United States was 3,026,789 square miles, or how the astronomer knows that the moon is 238,000 miles from the earth? In brief, do you know how to measure an object without measuring it? Such problems must be solved every day by engineers constructing bridges, large buildings and railroads, by astronomers in determining the time by which our clocks are regulated, and by a host of other people. This chapter shows how such measures may be made.

Problem. An observer in a lighthouse is 200 feet above sea level. He sees a ship at an angle of depression of 16°. How far is the ship from the foot of the lighthouse?

In order to solve problems of this kind, you must understand the ratios of the sides of a *right triangle* and their use in *trigonometry* or triangle measure. These ratios are called *trigonometric ratios.*

Let us recall the three trigonometric ratios which you have studied in algebra and geometry.

Recall fact 1. *In a right triangle the tangent of an acute angle is the fraction (ratio) whose numerator is the opposite leg and whose denominator is the adjacent leg.*



$$\text{tangent angle } A = \frac{\text{opposite leg}}{\text{adjacent leg}};$$

$$\text{or, briefly, } \tan A = \frac{a}{b}.$$

As the value of each trigonometric ratio depends upon, *i.e.*, is a function of, the size of the angle with which it is associated, the trigonometric ratios are also called *trigonometric functions.*

To make use of the tangent ratio in solving problems it is necessary to know its numerical values for angles from 0° to 90°. The table on page 3 gives these values. By means of this table we can find the value of the tangent of an angle and also the number of degrees in an angle whose tangent is known.

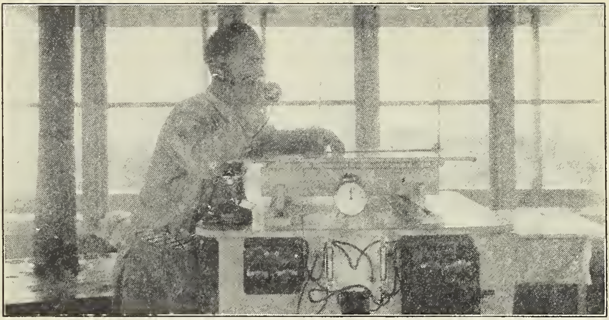
Thus, to find the value of tan 16°, find 16° in the column headed **Angle.** To the right of this number on the same horizontal line in the column headed **tan** you will find .2867, that is, tan 16° = .2867. Values of the other trigonometric functions of 16° are found in the other columns.

For convenience, the table is arranged so that the angles from 0° to 45° are in the left-hand column and the functions of these angles are in the six columns at the right. The names of the functions appear at the top. The angles from 45° to 90° are in the right-hand column and the functions of these angles are in the six columns at the left. The names of the functions appear at the bottom.

Thus, to find the angle to the nearest degree whose tangent is equal to 1.2950, first find in one of the tangent columns the number nearest to 1.2950. This number is 1.2799. To the right of this number in the angle column (since 1.2799 is in the column labeled **tan** at the bottom) you will find 52°. That is, the angle whose tangent is 1.2950 is 52°.

VALUES OF THE TRIGONOMETRIC FUNCTIONS

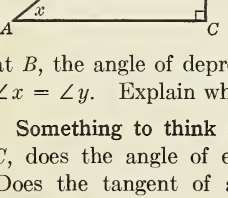
Angle	sin	cos	tan	cot	sec	csc	
0°	.0000	1.0000	.0000	∞	1.0000	∞	**90°**
1°	.0175	.9998	.0175	57.2900	1.0002	57.2987	89°
2°	.0349	.9994	.0349	28.6363	1.0006	28.6537	88°
3°	.0523	.9986	.0524	19.0811	1.0014	19.1073	87°
4°	.0698	.9976	.0699	14.3007	1.0024	14.3356	86°
5°	.0872	.9962	.0875	11.4301	1.0038	11.4737	**85°**
6°	.1045	.9945	.1051	9.5144	1.0055	9.5668	84°
7°	.1219	.9925	.1228	8.1443	1.0075	8.2055	83°
8°	.1392	.9903	.1405	7.1154	1.0098	7.1853	82°
9°	.1564	.9877	.1584	6.3138	1.0125	6.3925	81°
10°	.1736	.9848	.1763	5.6713	1.0154	5.7588	**80°**
11°	.1908	.9816	.1944	5.1446	1.0187	5.2408	79°
12°	.2079	.9781	.2126	4.7046	1.0223	4.8097	78°
13°	.2250	.9744	.2309	4.3315	1.0263	4.4454	77°
14°	.2419	.9703	.2493	4.0108	1.0306	4.1336	76°
15°	.2588	.9659	.2679	3.7321	1.0353	3.8637	**75°**
16°	.2756	.9613	.2867	3.4874	1.0403	3.6280	74°
17°	.2924	.9563	.3057	3.2709	1.0457	3.4203	73°
18°	.3090	.9511	.3249	3.0777	1.0515	3.2361	72°
19°	.3256	.9455	.3443	2.9042	1.0576	3.0716	71°
20°	.3420	.9397	.3640	2.7475	1.0642	2.9238	**70°**
21°	.3584	.9336	.3839	2.6051	1.0711	2.7904	69°
22°	.3746	.9272	.4040	2.4751	1.0785	2.6695	68°
23°	.3907	.9205	.4245	2.3559	1.0864	2.5593	67°
24°	.4067	.9135	.4452	2.2460	1.0946	2.4586	66°
25°	.4226	.9063	.4663	2.1445	1.1034	2.3662	**65°**
26°	.4384	.8988	.4877	2.0503	1.1126	2.2812	64°
27°	.4540	.8910	.5095	1.9626	1.1223	2.2027	63°
28°	.4695	.8829	.5317	1.8807	1.1326	2.1301	62°
29°	.4848	.8746	.5543	1.8040	1.1434	2.0627	61°
30°	.5000	.8660	.5774	1.7321	1.1547	2.0000	**60°**
31°	.5150	.8572	.6009	1.6643	1.1666	1.9416	59°
32°	.5299	.8480	.6249	1.6003	1.1792	1.8871	58°
33°	.5446	.8387	.6494	1.5399	1.1924	1.8361	57°
34°	.5592	.8290	.6745	1.4826	1.2062	1.7883	56°
35°	.5736	.8192	.7002	1.4281	1.2208	1.7434	**55°**
36°	.5878	.8090	.7265	1.3764	1.2361	1.7013	54°
37°	.6018	.7986	.7536	1.3270	1.2521	1.6616	53°
38°	.6157	.7880	.7813	1.2799	1.2690	1.6243	52°
39°	.6293	.7771	.8098	1.2349	1.2868	1.5890	51°
40°	.6428	.7660	.8391	1.1918	1.3054	1.5557	**50°**
41°	.6561	.7547	.8693	1.1504	1.3250	1.5243	49°
42°	.6691	.7431	.9004	1.1106	1.3456	1.4945	48°
43°	.6820	.7314	.9325	1.0724	1.3673	1.4663	47°
44°	.6947	.7193	.9657	1.0355	1.3902	1.4396	46°
45°	.7071	.7071	1.0000	1.0000	1.4142	1.4142	**45°**
	cos	sin	cot	tan	csc	sec	Angle



Courtesy U. S. Forest Service

Explain how a ranger in the lookout station such as this one in Ouachita National Forest, Arkansas, can use angles of depression to locate forest fires.

Recall fact 2. *The angle of elevation of an object above the level of the eyes is the angle which a line to the object from the observer's eye (called the line of sight) makes with a horizontal line, both lines lying in the same vertical plane.*



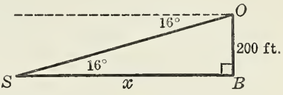
The angle of depression of an object below the level of the eye is the angle which the line of sight makes with a horizontal line, both lines lying in the same vertical plane.

Thus, standing at A, the angle of elevation of point B is $\angle x$. Standing at B, the angle of depression of point A is $\angle y$. Observe that $\angle x = \angle y$. Explain why.

Something to think about. As A moves along AC toward C, does the angle of elevation increase or decrease? Why? Does the tangent of angle A increase or decrease? Why? Does the angle of depression at B increase or decrease? Why? Does the tangent of angle B increase or decrease? Why?

Illustrative example. An observer in a lighthouse is 200 feet above sea level. He sees a ship at an angle of depression of 16°. How far is the ship from the foot of the lighthouse?

Given	*To find*
Right $\triangle OBS$.	$SB = ?$ ft.
$OB = 200$ ft.	
$\angle S = 16°$.	



Solution

$$\tan S = \frac{OB}{SB}.$$

$$\tan 16° = \frac{200}{x}.$$

$$.2867 = \frac{200}{x}.$$

$$.2867\,x = 200.$$

$$x = \frac{200}{.2867}.$$

$$x = 697.5.$$

Check

$$\tan 16° = \frac{200}{698}.$$

$$698 \tan 16° = 200.$$

$$698 \times .2867 = 200.$$

$$200 = 200.$$

The ship is about 698 feet from the foot of the lighthouse.

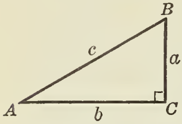
EXERCISES

1. A ladder that leans against the side of a building stands on level ground and makes an angle of 72° with the ground. If the foot of the ladder is 9 feet from the building, to what height on the building does the ladder reach?

2. At a point 55 feet from the base of a flagpole, standing on level ground, the angle of elevation of the top of the pole is 50°. Find the height of the flagpole, correct to the nearest foot.

3. A 3-foot pole casts a shadow 2 feet in length. What is the angle of elevation of the sun?

Recall fact 3. *In a right triangle the sine of an acute angle is the fraction (ratio) whose numerator is the opposite leg and whose denominator is the hypotenuse.*



$$\text{sine angle } A = \frac{\text{opposite leg}}{\text{hypotenuse}};$$

$$\text{or } \sin A = \frac{a}{c}.$$

Illustrative example. The length of a kite string is 225 feet. Assuming that the string lies in a straight line and that it makes an angle of 72° with the ground, find how high the kite is, correct to the nearest foot.

Given	*To find*
Right $\triangle BKC$. $BK = 225$ ft. $\angle B = 72°$.	$KC = ?$ ft.

Solution

$$\sin B = \frac{KC}{KB}.$$

$$\sin 72° = \frac{x}{225}.$$

$$.9511 = \frac{x}{225}.$$

$$x = 225 \times .9511 = 214.0 \text{ ft.}$$

The kite is 214 feet above the ground.

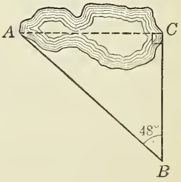
Check

$$\sin 72° = \frac{214}{225}.$$

$$.9511 = .9511.$$

EXERCISES

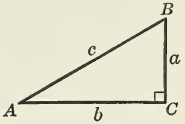
1. To find the length of a pond, an engineer stood at C and sighted to A. He turned at right angles to AC and walked to point B from which he found $\angle B$ to be 48°; AB was found to be 72 feet. How long was the pond?



Recall fact 4. *In a right triangle, the cosine of an acute angle is the fraction (ratio) whose numerator is the adjacent leg and whose denominator is the hypotenuse.*

$$\text{cosine angle } A = \frac{\text{adjacent leg}}{\text{hypotenuse}};$$

$$\text{or } \cos A = \frac{b}{c}.$$



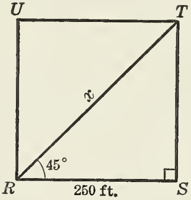
Illustrative example. Find correct to the nearest foot the length of a foot path running diagonally across a square lot whose side is 250 feet.

Given

Square $RSTU$.
$RS = 250$ ft.

To find

$RT = ?$ ft.



Solution

Since RT bisects $\angle URS$,
$\angle TRS = 45°$.

$$\cos TRS = \frac{RS}{RT}.$$

$$\cos 45° = \frac{250}{x}.$$

$$.7071\, x = 250.$$

$$x = \frac{250}{.7071} = 353.5.$$

Check

In any right triangle,

$$a^2 + b^2 = c^2.$$

$$(RS)^2 + (ST)^2 = (RT)^2.$$

$$(250)^2 + (250)^2 = (354)^2.$$

$$62{,}500 + 62{,}500 = 125{,}316.$$

$$125{,}000 = 125{,}316 \text{ (approximately)}.$$

The length of the foot path is 354 feet.

For this problem only an approximate check can be obtained. This is often true in problems of this kind because (1) the table gives approximate and not exact values for the functions, and (2) we used 354 feet and not 353.5 feet in the check.

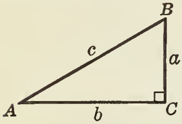
EXERCISES

1. Joseph placed an 18-foot ladder against a wall. If the ladder makes an angle of 56° with the floor, how far from the wall is the foot of the ladder?

2. The hypotenuse of a right triangle is 25 feet and one leg is 13 feet. Find the angle made by this leg and the hypotenuse correct to the nearest degree.

Discovering new ratios. In addition to the three ratios you have already studied, three other ratios may be obtained for each acute angle in any right triangle. With these ratios you can solve certain problems more easily than with the tangent, sine, and cosine.

Cotangent. *In a right triangle the cotangent of an acute angle is the ratio of the adjacent leg to the opposite leg.*



$$\text{cotangent angle } A = \frac{\text{adjacent leg}}{\text{opposite leg}};$$

$$\text{or cot } A = \frac{b}{a}.$$

Using the figure above, complete the following:

1. $\tan A = \frac{a}{?}$; $\cot A = \frac{?}{a}$; $\tan A \cdot \cot A = ?$

2. $\tan B = ?$; $\cot B = ?$; $\tan B \cdot \cot B = ?$

3. Show that $\tan 45° \cdot \cot 45° = 1$, by using the table on page 3.

4. The tangent of an angle times the cotangent of the same angle is always equal to ? . The cotangent is the reciprocal of the ? .

5. (*a*) If $\tan A = \frac{1}{2}$, then $\cot A = ?$
   (*b*) If $\tan A = 1$, then $\cot A = ?$
   (*c*) If $\tan A = \frac{3}{2}$, then $\cot A = ?$
   (*d*) If $\tan A = 2.4$, then $\cot A = ?$

6. If tan A increases, cot A ? . Check your conclusion with the values in the table.

7. If tan A = cot A, the right triangle is ? .

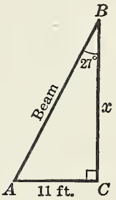
8. If the angle at A increases, cot A ? .

9. If the angle at A decreases, tan A ? .

10. If the angle at A increases, tan A ? .

Illustrative example. A beam supporting a fence makes an angle of 27° with it, and the foot of the beam is 11 feet from the fence. How high does the beam reach on the fence?

Given	*To find*
Right $\triangle ABC$. $\angle B = 27°$. $AC = 11$ ft.	BC = ? ft.



Solution	*Check*
$\cot B = \frac{BC}{AC}$. $\cot 27° = \frac{x}{11}$. $1.9626 = \frac{x}{11}$. $x = 21.6$.	$\angle A = 90° - \angle B$. $\angle A = 90° - 27° = 63°$. $\tan 63° = \frac{22}{11}$. $1.9626 = 2$ (approximately).

The beam reaches a point on the fence about 22 feet from the ground.

EXERCISES

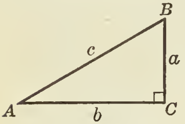
1. A vertical pole 12 feet high casts a shadow of 15.6 feet. What is the angle of elevation of the sun?

2. From 1500 feet above a trench, an observer in an airplane finds that the angle of depression of an enemy fort is 29°. How far is the trench from the fort?

3. From a certain point in Washington the angle of elevation of the top of the Washington Monument (555 feet high) is 30°. How far is this point from the base of the monument?

4. If a rope is attached to the top of a 25-foot pole, how far from the foot of the pole must the rope be fastened if it is to make an angle of 37° with the ground?

Secant. *In a right triangle the secant of an acute angle is the ratio of the hypotenuse to the adjacent leg.*



$$\text{secant angle } A = \frac{\text{hypotenuse}}{\text{adjacent leg}};$$

$$\text{or sec } A = \frac{c}{b}.$$

Using the figure shown above, complete the following:

1. $\sec A = \frac{c}{?}$; $\cos A = \frac{?}{c}$; $\sec A \cdot \cos A = ?$

2. $\sec B = ?$; $\cos B = ?$; $\sec B \cdot \cos B = ?$

3. The secant of an angle times the cosine of the same angle is always equal to ? The secant is the ? of the cosine.

4. By using the table on page 3, show that:
(*a*) $\sec 43° \cdot \cos 43° = 1$. (*b*) $\cot 17° \cdot \tan 17° = 1$.

5. (*a*) If $\cos A = \frac{1}{4}$, then $\sec A = ?$
(*b*) If $\cos A = \frac{1}{3}$, then $\sec A = ?$
(*c*) If $\cos A = \frac{1}{2}$, then $\sec A = ?$
(*d*) If $\cos A = 0.9$, then $\sec A = ?$

6. If $\cos A$ increases, $\sec A$? . If $\sec A$ increases, $\cos A$? . Check your answers with the values in the table.

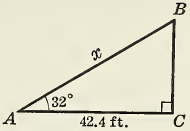
7. From the definition of a secant it is evident that its numerical value is greater than ? . Why? The numerical value of a cosine must therefore be less than ? . Why?

8. If the angle at A increases, $\sec A$? .

9. If the angle at A decreases, $\sec A$? , while $\cos A$? .

Illustrative example. In right triangle ABC, $AC = 42.4$ feet and $\angle A = 32°$. How long is AB?

Given	*To find*
$AC = 42.4$ ft.	$AB = ?$ ft.
$\angle A = 32°$.	



Solution

$$\sec A = \frac{AB}{AC}.$$

$$\sec 32° = \frac{x}{42.4}.$$

$$1.1792 = \frac{x}{42.4}.$$

$$x = 50\ 00.$$

AB is about 50 feet long.

Check

$$\cos 32° = \frac{42.4}{50}.$$

$$.8480 = .8480.$$

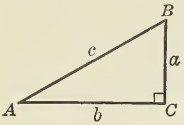
EXERCISES

1. The length of an auditorium floor is 300 feet and its width is 120 feet. The seating floor is inclined 10° to the original floor. What is the length of the seating floor? What would be the cost of cementing the seating floor at *a* cents per square foot?

2. An engineer is required to dig a vertical shaft to meet a tunnel into the earth which descends at an angle of 16°. If he commences to dig at a point on the ground directly above the tunnel and 275 feet from the entrance, how far from the entrance will the shaft meet the tunnel?

3. In building a suspension bridge a straight cable is to run from the top of the pier to a point 800 ft. from the foot of the pier. If from this point the angle of elevation of the top of the pier is 27°, what length of cable will be needed?

Cosecant. *In a right triangle the cosecant of an acute angle is the ratio of the hypotenuse to the opposite leg.*



$$\text{cosecant angle } A = \frac{\text{hypotenuse}}{\text{opposite leg}};$$

$$\text{or } \csc A = \frac{c}{a}.$$

Using the adjoining figure, copy and complete the following:

1. $\csc A = \dfrac{c}{?}$; $\sin A = \dfrac{?}{c}$; $\csc A \cdot \sin A = ?$

2. $\csc B = ?$; $\sin B = ?$; $\csc B \cdot \sin B = ?$

3. The cosecant of an angle times the sine of the same angle is always equal to ? . The cosecant is the reciprocal of the ? .

4. By using the table on page 3, show that:

(*a*) $\csc 37° \cdot \sin 37° = 1.$ (*b*) $\sec 20° \cdot \cos 20° = 1.$
(*c*) $\cot 70° \cdot \tan 70° = 1.$

5. (*a*) If $\sin A = \frac{1}{6}$, then $\csc A = ?$
(*b*) If $\sin A = \frac{1}{3}$, then $\csc A = ?$
(*c*) If $\sin A = .7$, then $\csc A = ?$
(*d*) If $\sin A$, *i.e.*, $\dfrac{a}{c}$, < 1, show that $\csc A$, *i.e.*, $\dfrac{c}{a}$, > 1.

6. If $\sin A$ increases, $\csc A$? . Check with the table.

7. From the definition of the cosecant it is evident that its numerical value is greater than ? . Why? The numerical value of the sine must therefore be less than ? . Why?

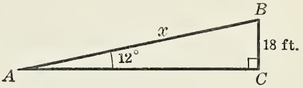
8. If the angle at A in the figure above increases, $\csc A$? . If the angle at A decreases, $\csc A$? , while $\sin A$? .

9. In $\triangle ABC$, $\angle A = 42°$. Which two ratios could be used in solving each of the following exercises and which involves the less work:

(*a*) $b = 100$, $c = ?$ (*c*) $a = 100$, $b = ?$
(*b*) $c = 100$, $b = ?$ (*d*) $b = 100$, $a = ?$

Illustrative example. A street up the side of a hill is inclined 12° to a horizontal line. How many feet must a boy walk up the hill to reach an altitude of 18 feet above the base of the hill?

Given	*To find*
Right $\triangle ABC$.	$AB = ?$ ft.
$\angle A = 12°$.	
$BC = 18$ ft.	



Solution	*Check*
$\csc A = \dfrac{AB}{BC}$.	$\sin A = \dfrac{BC}{AB}$.
$\csc 12° = \dfrac{x}{18}$.	$\sin 12° = \dfrac{18}{87}$.
$4.8097 = \dfrac{x}{18}$.	$.2079 = .2069$ (approximately).
$x = 86.6$.	

The boy must walk about 87 feet.

EXERCISES

1. One leg of a right triangle is 24 inches and the opposite angle is 36°. Find the length of the hypotenuse correct to the nearest inch.

2. A pole 7 feet long is supported in a vertical position by ropes 25 feet long running from the top of the pole to stakes in the ground. At what angle is each rope inclined to the ground?

Remember

$$\sin A \cdot \csc A = 1 \quad \text{or} \quad \csc A = \frac{1}{\sin A}.$$

$$\cos A \cdot \sec A = 1 \quad \text{or} \quad \sec A = \frac{1}{\cos A}.$$

$$\tan A \cdot \cot A = 1 \quad \text{or} \quad \cot A = \frac{1}{\tan A}.$$

GENERAL EXERCISES

In solving these problems, use the trigonometric function which will make your work easiest. A diagram will help you to solve each problem correctly.

Group A

1. A and C are two points 115 feet apart on level ground; B is a balloon directly above C; from A the angle of elevation of the balloon is 53°. Find the height of the balloon correct to the nearest foot.

2. From the foot of a tower a Boy Scout measures a horizontal line 120 feet long and at its end finds the angle of elevation to the top of the tower to be 48°. What is the height of the tower?

3. A boy measured the angle of elevation of the top of a flag staff whose height he knew to be 160 feet, and found it to be 20°. How far was he from the bottom of the staff?

4. A 3-foot stick held vertically casts a shadow 1 foot in length. Find the angle of elevation of the sun.

5. Two sides of a triangle are 16 inches and 20 inches and the angle between them is 27°. Find the altitude upon the 20-inch side correct to the nearest inch. Find the area of the triangle correct to the nearest square inch.

Group B

6. A vertical pole 80 feet high stands at the center of a circular pond. At a point in the circumference the angle of elevation of the top of the pole is 50°. Find the radius of the pond correct to the nearest foot.

7. The diagonal of a rectangle is 40 inches long. It makes an angle of 40° with the longer side. Find the length, the width, and the area of the rectangle correct to the nearest unit.

8. The altitude of an equilateral triangle is 8.3 inches. Find the perimeter of the triangle.

9. City C is 60 miles east of city B and 39 miles south of city A. Find the number of degrees in the angle made by lines running from city B to cities C and A.

10. From the base of a window 35 feet above street level, the angle of depression of the base of a building across the street is 27° and the angle of elevation of the top of the same building is 63°. Find the height of the building.

Group C

11. A man observes the angle of elevation of the top of a tower to be 30°. He walks toward the tower for 300 feet and then finds the angle of elevation to be 60°. How high is the tower?

12. From the top of a lighthouse 260 feet above sea level, the angles of depression to two boats in line with the lighthouse are observed to be 16° and 32° respectively. Find the distance between the two boats.

13. From the top of a hill b feet high, the angles of depression of two ships in line with the foot of the hill are $m°$ and $q°$ respectively. Find the distance between the ships in terms of b, m, and q.

14. The angle of elevation of the top of a tower from a point P on the ground is 60°. The angle of depression of the top of the tower from an airplane 400 feet directly above P is 30°. Find the height of the tower.

15. From a point on the ground the angle of elevation of a balloon is 28°. After the balloon ascends vertically a distance of 175 feet, the angle of elevation from the same point is 44°. Find the height of the balloon in the last position.

16. Make a formula for solving problems like Ex. 15, using $x°$ in place of 28°, a ft. for 175 ft. and $y°$ for 44°.

To Construct an Angle from a Known Trigonometric Function

The following illustration will show you how to construct an angle when one of its trigonometric functions is known and also how to find the values of the other five functions.

Illustrative example. The sine of angle A equals two-thirds. Construct the angle and find the values of the other functions.

Given	*Required*
$\sin A = \frac{2}{3}$.	1. To construct $\angle A$. 2. To find the other functions.

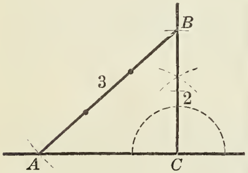
Analysis

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} \text{ and } \sin A = \tfrac{2}{3}. \quad \therefore \frac{\text{opposite}}{\text{hypotenuse}} = \frac{2}{3}.$$

Hence we are to build a right triangle whose hypotenuse is 3 and one leg is 2. The $\angle$ opposite 2 will be $\angle A$.

Solution

Construct a perpendicular to AC at C. On CB measure off a distance $CB = 2$. With B as center and 3 as radius, cut AC at A. ABC is the required triangle.



$$c^2 = a^2 + b^2.$$
$$3^2 = 2^2 + b^2.$$
$$b = \sqrt{5}.$$

$$\sin A = \tfrac{2}{3}.$$
$$\cos A = \frac{\sqrt{5}}{3}.$$
$$\tan A = \frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}.$$
$$\cot A = \frac{\sqrt{5}}{2}.$$
$$\sec A = \frac{3}{\sqrt{5}} = \frac{3}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{5}.$$
$$\csc A = \tfrac{3}{2}.$$

EXERCISES

Group A

Construct the angle and write the values of the remaining functions if:

1. $\tan A = \frac{5}{12}$.
2. $\sin A = \frac{3}{5}$.
3. $\cos A = \frac{8}{17}$.
4. $\cot A = \frac{4}{3}$.
5. $\tan B = 1$.
6. $\csc A = \frac{7}{5}$.

7. Is it possible to construct an angle such that the sine of the angle equals $\frac{5}{4}$? Why?

8. Determine by construction whether doubling an angle also doubles the value of its sine. Check your conclusion by the values in the table.

Group B

Construct the angle if:

9. $\tan x = .75$.

10. $\cos y = \dfrac{\sqrt{2}}{2}$.

11. $\sin x = \dfrac{\sqrt{3}}{2}$.

12. $\sin x = \cos x$.
$\left(\text{As } \dfrac{a}{c} = \dfrac{b}{c}, a \text{ must equal } b. \quad \text{Why?}\right)$

13. $\sin y = 2 \cos y$.

14. $4 \sin y = \tan y$.

Group C

15. Which is greater: $\sin A$ or $\tan A$? $\sin A$ or $\sec A$? Why?

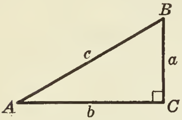
16. Given an acute angle x, construct an angle y such that:
(*a*) $\tan y = 2 \tan x$. (*b*) $\sin y = 2 \sin x$.

17. As an angle increases from $0°$ to $90°$, which function increases the more rapidly, the sine or the tangent? Why?

18. If an angle is bisected, is its tangent bisected also? Illustrate by a diagram. Check, using the values in the table on page 3.

Co-functions. The following will help you to discover the relation between the functions of the acute angles of a right triangle.

1. Complete each of the following from the adjoining figure. What relation do you find between the values of the functions of $\angle A$ and $\angle B$?



$\sin A = \frac{a}{?}.$	$\sin B = \frac{b}{?}.$
$\cos A = \frac{?}{c}.$	$\cos B = \frac{?}{c}.$
$\tan A = ?$	$\tan B = ?$
$\cot A = ?$	$\cot B = ?$
$\sec A = ?$	$\sec B = ?$
$\csc A = ?$	$\csc B = ?$

2. As the sum of the acute angles in any right triangle is always ? degrees, the acute angles are said to be ? .

3. As the sine and cosine ratios are formed in like manner (ratio of leg to hypotenuse) each is called a co-function of the other. Similarly the tangent is the co-function of the ? . The secant and cosecant are ? .

4. $\sin A = \cos B.$ $\qquad$ $\cot B = \ ?\ A.$
$\cos A = \ ?\ B.$ $\qquad$ $\sec B = \ ?\ A.$
$\tan A = \ ?\ B.$ $\qquad$ $\csc B = \ ?\ A.$

We thus discover that: *The function of any acute angle is equal to the co-function of its complementary angle.*

Historically " co-function " means " complement's function." Originally, " cosine " was written " complement's sine " ; later it was abbreviated to " co-sine " and finally took the modern form " cosine."

Something to think about. Why can a table of functions of angles from 0° to 45° inclusive be used to give the values of functions of angles from 45° to 90°?

EXERCISES

1. Express as functions of complementary angles:

cos 20°.	sin 40°.	cot 10° 30′.
sin 45°.	sec 3°.	tan 50° 40′.
tan 88°.	csc 72°.	sin 0° 10′.
sec 62°.	tan 36°.	csc 83° 14′.
cot 45°.	sin 8°.	cos x°.

2. Express as functions of angles less than 45°:

sin 75°.	cos 46°.	tan 89°.	cot 70° 17′.
cot 61°.	sec 70°.	csc 53°.	sin 45° 16′.
tan 50°.	sin 82°.	sec 54°.	cos 80° 47′.

3. (*a*) If $\sin 60° = \frac{1}{2}\sqrt{3}$, then $\cos 30° = ?$
(*b*) If $\cos 45° = \frac{1}{2}\sqrt{2}$, then $\sin 45° = ?$
(*c*) If $\sec 30° = \frac{2}{3}\sqrt{3}$, then $\csc 60° = ?$
(*d*) If $\tan 45° = 1$, then $\cot 45° = ?$
(*e*) If $\cot 30° = \sqrt{3}$, then $\tan 60° = ?$

4. If a function of one acute angle equals a co-function of another acute angle, the sum of the angles is 90°. Why? Using this fact, find the number of degrees in $\angle A$ if:

(*a*) $\sin A = \cos A$.
(*b*) $\sin 2A = \cos A$.
(*c*) $\tan 2A = \cot 3A$.
(*d*) $\cos \frac{1}{2} A = \sin A$.
(*e*) $\sec 2A = \csc 7A$.
(*f*) $\sin 2A = \cos (45° + A)$

5. If A and B are the acute angles of a right triangle, express the functions of A as functions of B.

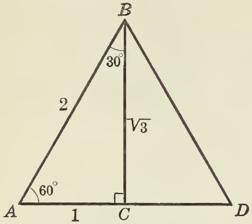
6. Write the value of all of the functions of $\angle B$ in right triangle ABC, if:

(*a*) $\sin A = \frac{4}{5}$.
(*b*) $\cos A = \frac{5}{13}$.
(*c*) $\tan A = .75$.
(*d*) $\cot A = 1$.
(*e*) $\sec A = 2$.
(*f*) $\csc A = k$.

Functions of 0°, 30°, 45°, 60°, and 90°

The values of the functions of 0°, 30°, 45°, 60°, and 90° are used so often that you should know how to find them without using the tables.

Functions of 30° and 60°. In this figure, triangle ABD is equilateral and BC is perpendicular to AD.



Therefore, $\angle A = 60°$;

$\angle ABC = 30°$;

also $AC = \frac{1}{2} AD$ or $\frac{1}{2} AB$.

If we let $AB = 2$, then $AC = 1$ and $BC = \sqrt{2^2 - 1^2} = \sqrt{3}$. From the figure we can tell the values of the functions:

$\sin 30° = \frac{1}{2}$. $\qquad$ $\sin 60° = \frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}$.

$\cos 30° = \frac{\sqrt{3}}{2}$. $\qquad$ $\cos 60° = \frac{1}{2}$.

$\tan 30° = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$. $\qquad$ $\tan 60° = \sqrt{3}$.

$\cot 30° = ?$ $\qquad$ $\cot 60° = ?$

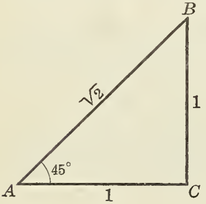
$\sec 30° = ?$ $\qquad$ $\sec 60° = ?$

$\csc 30° = ?$ $\qquad$ $\csc 60° = ?$

Something to think about. Why did we let AB equal 2 rather than 1 or any other value? If we had selected another value for AB, would we have obtained the same values for the functions of 30° and 60°? Test your answer by letting $AB = 4$.

Functions of 45°. In the adjoining figure, $\triangle ABC$ is an isosceles right triangle. Therefore, $\angle A = 45°$ and $\angle B = 45°$.

If we let $AC = 1$, then $BC = 1$ and $AB = \sqrt{2}$. From the figure we can tell the values of the functions:



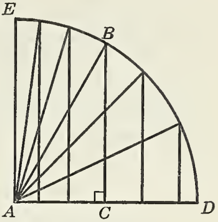
$\sin 45° = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$.

$\cos 45° = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$.

$\tan 45° = 1$. $\qquad$ $\sec 45° = ?$

$\cot 45° = ?$ $\qquad$ $\csc 45° = ?$

Functions of 0° and 90°. So far we have dealt only with the functions of acute angles in a right triangle. When we consider angles of 0° and 90°, we cannot derive the values of their functions from a right triangle. In the diagram adjoining there are several right triangles in a quadrant of a circle whose radius is 1. By means of this diagram, complete the following by using: *increases, decreases, remains constant.*



1. As B moves downward on the quadrant:
   (a) $\angle A$? ; BC ? ; AC ? ; AB ? .
   (b) $\frac{BC}{AB}$? ; $\frac{AC}{AB}$? ; $\frac{BC}{AC}$? .
   (c) $\angle A$? ; sin A ? ; cos A ? ; tan A ? .

Now let us tabulate what happens as B approaches and finally reaches D.

	Change	Approaches	And finally becomes
As point B	moves downward	point D	point D
line BC	decreases	0	0
line AC	increases	AD or 1	1
AB or 1	unchanged	—	—
angle A	decreases	0°	0°
sin A $\left(\frac{BC}{AB}\right)$	decreases	$\frac{0}{1}$	0
cos A $\left(\frac{AC}{AB}\right)$	increases	$\frac{1}{1}$	1
tan A $\left(\frac{BC}{AC}\right)$	decreases	$\frac{0}{1}$	0

Remember

sin 0° = 0; cos 0° = 1; tan 0° = 0.

2. As B moves upward on the quadrant:

(a) $\angle A$? ; BC ? ; AC ? ; AB ? .

(b) $\frac{BC}{AB}$? ; $\frac{AC}{AB}$? ; $\frac{BC}{AC}$? .

(c) $\angle A$? ; $\sin A$? ; $\cos A$? ; $\tan A$? .

Now let us tabulate what happens as B approaches and finally reaches E.

	Change	Approaches	And finally becomes
As point B	moves upward	point E	point E
line BC	increases	AE or 1	1
line AC	decreases	0	0
AB or 1	unchanged	—	—
angle A	increases	90°	90°
$\sin A\left(\frac{BC}{AB}\right)$	increases	$\frac{1}{1}$	1
$\cos A\left(\frac{AC}{AB}\right)$	decreases	$\frac{0}{1}$	0
$\tan A\left(\frac{BC}{AC}\right)$	increases	$\frac{1}{0}$	$\frac{1}{0}$

But what is the value of $\frac{1}{0}$? The answers to the following questions will help you decide.

3. $\frac{1}{1}=?$ $\quad \frac{1}{.1}=?$ $\quad \frac{1}{.01}=?$ $\quad \frac{1}{.001}=?$ $\quad \frac{1}{.000001}=?$

4. $\frac{5}{1}=?$ $\quad \frac{5}{.1}=?$ $\quad \frac{5}{.01}=?$ $\quad \frac{5}{.001}=?$ $\quad \frac{5}{.000001}=?$

5. $\frac{a}{1}=?$ $\quad \frac{a}{.1}=?$ $\quad \frac{a}{.01}=?$ $\quad \frac{a}{.001}=?$ $\quad \frac{a}{.000001}=?$

6. If the numerator (not zero) is constant while the denominator decreases, does the quotient increase or decrease?

7. If the numerator (not zero) is constant, the smaller the denominator, the ? the quotient.

From the foregoing we may conclude that if the numerator (not zero) is constant while the denominator approaches 0 and finally becomes 0, the quotient grows very large and finally becomes greater than any number. Thus the result of dividing 1 or any number (not zero) by 0 is larger than any known number. We call this result *infinity* (written ∞).

Remember

$\sin 90° = 1$; $\cos 90° = 0$; $\tan 90° = \infty$.

Complete the following, using the figure on page 21:

$\cot 0° = ?$ $\sec 0° = ?$ $\csc 0° = ?$

$\cot 90° = ?$ $\sec 90° = ?$ $\csc 90° = ?$

Remember

ANGLE	0°	30°	45°	60°	90°
sin	$\frac{1}{2}\sqrt{0}$ or 0	$\frac{1}{2}\sqrt{1}$ or $\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{4}$ or 1
cos	$\frac{1}{2}\sqrt{4}$ or 1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{1}$ or $\frac{1}{2}$	$\frac{1}{2}\sqrt{0}$ or 0
tan	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	∞

EXERCISES

Group A

1. The sides of a right triangle are 3, 4, and 5 respectively. Write the six functions of the smallest angle.

2. The sides of a right triangle are 5, 12, and 13 respectively. Write the six functions of each acute angle.

3. The legs of a right triangle are 6 and 8 respectively. Find the hypotenuse and the six functions of the smallest angle.

4. Find the numerical value of:

(*a*) $\sin 60° + \cos 30°$. (*c*) $\tan 45° + \cot 45°$.

(*b*) $\cos 60° + \sin 30°$. (*d*) $2 \sin 30° - \frac{1}{2} \cos 60°$.

5. What is the value of the following ratios:

(*a*) $\tan 45° : \cot 45°$. (*c*) $\cos 60° : \sin 30°$.
(*b*) $\sin 60° : \cos 30°$. (*d*) $\sin 45° : \sin 30°$.

6. As an angle increases from 0° to 90°, which functions increase in value? Which functions decrease in value?

7. Find the number of degrees in angle x (*i.e.*, solve the equation) if:

(*a*) $\sin x = \frac{1}{2}$. (*d*) $\tan x = 1$.
(*b*) $\sin x = \frac{1}{2}\sqrt{2}$. (*e*) $\sin x = 1$.
(*c*) $\cos x = \frac{1}{2}\sqrt{3}$. (*f*) $\cos x = 0$.

Group B

Show whether:

8. (*a*) $\sin 60° = 2 \sin 30°$.
(*b*) $\tan 30° = \frac{1}{2} \tan 60°$.
(*c*) $\cos 60° = \cos 30° + \cos 30°$.

9. Find the value of:

(*a*) $(\sin 60° + \cos 30°) \sin 90°$.
(*b*) $(\cos 60° + \sin 30°) \cos 90°$.
(*c*) $(\tan 45° - \cot 45°) \sin 30°$.
(*d*) $(2 \sin 30° - \frac{1}{2} \cos 60°) \cos 60°$.

10. The sides of a right triangle are in the ratio of 8 : 15 : 17. Find the six functions of each acute angle.

11. The legs of a right triangle are 7 and 24. Find the hypotenuse and the six functions of the smallest angle.

12. Find the value of $(8 \tan 0° - \cot 90° + \sin 30°) \cos 90°$.

13. Which is greater:

(*a*) $3 \sin 30°$ or $\sin (3 \times 30°)$?
(*b*) $2 \tan 45°$ or $\tan (2 \times 45°)$?

14. What value of x satisfies the equation:

(*a*) $\sin x = \cos x$? (*c*) $(\sin x)^2 = \frac{3}{4}$?
(*b*) $\tan x + \cot x = 2$? (*d*) $(\cos x)^2 = \frac{1}{2}$?

Group C

15. If $A = 90°$, $B = 60°$, $C = 30°$, $D = 45°$, and $E = 0°$, show that:

(*a*) $2 \sin D \cos D = \sin A \cos E$.

(*b*) $2 \sin C \cos C = \sin B$.

(*c*) $\sin^2 B + \cos^2 B = 1$. ($\sin^2 B = \sin B \cdot \sin B$.)

16. The sides of a right triangle are in the ratio of $1 : 2 : \sqrt{3}$. Find the six functions of each acute angle.

17. The two legs of a right triangle are m and n. Find the hypotenuse and the functions of one of the acute angles.

18. If $\tan A = 4$, find the value of $\sin A \sec A$.

19. Find the value of $(x - y)^2 \sin 60° - (x - y)^2 \cos 30°$.

20. By how much does $\sin 90°$ exceed $\sin 30°$? $\cos 30°$ exceed $\cos 60°$?

21. Is $\sin (30° + 45°) = \sin 75°$? Why?
Is $\sin 30° + \sin 45° = \sin 75°$? Why?

Forces

Composition of forces. A push or pull acting on a body is called a *force*. Forces are usually measured in pounds. When the pull is exerted through a cord or wire, it is called a *tension*.

Often there are several forces acting on the same body, and in order to determine their effect, it is necessary to find a single force that would produce the same result as their combined action on that body.

A single force that would have the same effect as two or more forces is their *resultant*, and the two or more forces are the *components* of the resultant. If two forces, acting on a point, be represented in amount and direction by the two sides of a parallelogram meeting at that point, the diagonal from that vertex will represent the resultant in amount and direction.

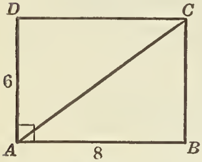
By means of a parallelogram two motions can be combined in the same way that two forces can.

Illustrative examples.

Example 1. Find the amount and direction of the resultant of two forces of 6 lb. and 8 lb. acting at right angles.

Solution

Let AB and AD be the two forces in amount and direction and complete the parallelogram $ABCD$. Then AC is the resultant.



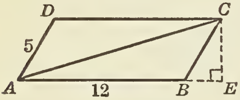
$$AC^2 = AB^2 + BC^2 = 64 + 36 = 100.$$

$$\therefore\ AC = 10 \text{ lb.}$$

$$\tan \angle CAB = \tfrac{6}{8} = .7500.$$

$$\angle CAB = 37^\circ \text{ approximately.}$$

Example 2. Find the amount and direction of the resultant of forces of 5 lb. and 12 lb. acting at an angle of 60° with each other.



Solution

Let AB and AD represent the two forces where $\angle DAB = 60^\circ$. Complete the parallelogram and draw $CE \perp AB$.

$$CE = BC \sin 60^\circ = 5 \times .8660 = 4.33.$$

$$BE = BC \cos 60^\circ = 5 \times .5000 = 2.50.$$

$$AE = AB + BE = 12 + 2.50 = 14.50.$$

$$AC^2 = CE^2 + AE^2 = 18.7489 + 210.25 = 228.9989.$$

$$AC = 15.13 = 15 \text{ lb. approximately.}$$

$$\tan \angle CAE = \frac{CE}{AE} = \frac{4.33}{14.50} = .2986.$$

$$\angle CAE = 17^\circ.$$

EXERCISES

Find the amount and direction of the resultant:

1. 30 lb. and 40 lb. acting at right angles.
2. 50 lb. and 60 lb. acting at an angle of 45°.
3. 100 lb. and 100 lb. acting 70° apart.
4. 1000 lb. and 950 lb. acting 150° apart.

Resolution of forces. A force may be considered as the resultant of the two forces represented by the sides of any parallelogram of which it is the diagonal. Practically, we generally use one component in the direction we are considering and one component at right angles to that. Our force then is the diagonal of a rectangle.

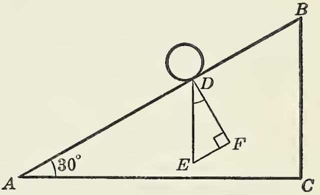
If we neglect friction, the only force that can act on a surface acts perpendicularly to that surface.

Illustrative example. Find the force needed to keep a barrel weighing 200 lb. from moving down a smooth surface inclined 30° to the horizontal.

Solution

Let DE represent the weight of the barrel, DF the component perpendicular to AB, FE the component parallel to AB, and $\angle BAC = 30°$.

Then $\angle EDF = 30°$.



$$FE = DE \sin 30° = 200 \times .5000 = 100 \text{ lb.}$$

EXERCISES

1. A boy is drawing his sled by pulling with a force of 15 lb. on the cord which makes an angle of 40° with the horizontal. What horizontal force is acting on the sled?

2. Mr. Strong is pushing with a force of 80 lb. on the handle of his lawn mower which makes an angle of 50° with the ground. What force does he exert parallel to the ground?

3. In Ex. 2, if Mr. Strong lowered the handle of his mower until it made a 35° angle with the ground, by how much would the push necessary to move the lawn mower be decreased?

4. John can push with a force of 60 lb. If he wishes to hold a 150-lb. barrel from rolling down an inclined plank, what is the greatest angle of elevation that the plank could have?

5. The wing of an airplane makes an angle of 10° with the direction of the wind. What force perpendicular to the wing is exerted by a wind force of 800 lb.?

6. The sail of a sailboat is set at an angle of 28° with the direction of the wind. What part of the force of the wind is acting perpendicular to the sail?

7. A hillside rises 52° from the horizontal. What force would be needed to drive a car weighing 2000 lb. along a road directly up the hill? What force is needed to drive it along a road that winds up a hill at an angle of 12° with the horizontal?

How Old Is Trigonometry?

Hipparchus (140 B.C.) is credited with having originated the science of trigonometry. He was interested in trigonometry, however, only in so far as it aided him in his astronomical work. Although none of his mathematical writings is extant, references made to his work by later scholars indicate that he was the first to compute tables resembling our tables of sines.

The first work in the construction of accurate tables of the trigonometric ratios was done by a group of German astronomers, the greatest of whom was Regiomontanus (1436–1476). His treatise *De triangulis omnimodis* (1464), marks the beginning of trigonometry as a separate subject. From that time on mathematicians made important discoveries and the subject of trigonometry has developed so that it is now linked up with every field of mathematics.

CUMULATIVE REVIEW

Chapter I

1. Which of these statements are true? Which are false?

(*a*) If angles A and B are acute and $A > B$, then $\sin A > \sin B$.

(*b*) $\cos 68° > \cos 48°$.

(*c*) As an angle increases from 0° to 90°, the tangent of the angle increases more rapidly than the sine.

(*d*) If $\sin A = .5000$, then $\sin 2A = 1.0000$.

(*e*) If angle A = angle B, any function of A is equal to the corresponding co-function of B.

2. Complete each of the following statements:

(*a*) As angle A increases from 0° to 90°, $\cos A$ decreases from ? to ? .

(*b*) As angle A increases from 0° to 90°, $\dfrac{\cos A}{\sin A}$? from ? to ? .

(*c*) As angle A increases from 0° to 90°, $\sin A$? from ? to ? .

(*d*) The sides of a right triangle are 28, 45, 53. The number of degrees in the smallest angle is ? (correct to the nearest degree).

(*e*) The number of degrees in the angle whose secant is equal to its cosecant is ? .

3. Find the value of: $\sin 60° \csc 60° - \sin^2 45° - \cos^2 45° + 2 \sin 90° \cos 0° - 2 \tan^2 30°$.

4. If $\sin A = \frac{5}{13}$, construct angle A and write the values of the other functions of A.

5. If $\sec B = \frac{5}{3}$, construct angle B and write the values of the other functions of B.

6. The two legs of a right triangle are 16 inches and 30 inches. Find the six functions of the smallest angle of the triangle.

7. A tunnel slopes downward at an angle of 12° with the ground. Assuming that the ground is level, how far below it is a worker in the tunnel 557 feet from the opening?

CHAPTER II. THE MEANING OF NUMBER AND THE FUNDAMENTAL PROCESSES

The invention or discovery of symbols is doubtless by far the single greatest event in the history of man. — John Dewey.

The Meaning of Number

In arithmetic you learned to use numbers greater than zero and numbers that contain zero. In elementary algebra you used these numbers and also numbers less than zero. A number greater than zero is a positive number, and one less than zero is a negative number. A collective name for positive and negative numbers is "signed numbers." If the sign before a number is plus it is a positive number, if the sign is minus it is a negative number.

Fundamental Processes

The basis of all mathematics is the meaning of number and the four fundamental processes — addition, subtraction, multiplication, and division. We will now briefly recall the basic facts of these processes as presented in elementary algebra.

Recall fact 5. *The absolute value of a positive or negative number is the arithmetical value of the number without regard to sign.*

The absolute value of -6 is six, of $+6$ is six. What is the absolute value of -9? Of $+12$?

Recall fact 6. *To add two signed numbers having like signs, add their absolute values and prefix their common sign.*

Recall fact 7. *To add two signed numbers having unlike signs, find the difference between their absolute values and prefix the sign of the number having the greater absolute value.*

EXERCISES

Add:

1.	+ 10	− 8	+ 12	− 16	+ 7	− .3
	+ 5	− 3	− 4	+ 11	− 13	+ .5
2.	+ 4	+ 16	+ 20	− 18	+ 6	− .7
	+ 7	− 8	− 15	− 3	− 5	− .1
	− 3	− 7	− 7	+ 21	+ 8	− .5
3.	+ 7	+ 5	− 3	− 8	− 9	− 3.4
	− 2	− 12	− 2	+ 4	+ 8	+ 2.8
	+ 3	− 3	+ 1	+ 3	− 7	− 0.6
	− 4	+ 5	+ 4	− 6	− 2	+ 1.3

Recall fact 8. *To subtract signed numbers: Mentally change the sign of the subtrahend (the quantity to be subtracted) and proceed as in algebraic addition.*

EXERCISES

Subtract:

+ 10	− 8	+ 4	+ 5	− 3	− 3	+ 4	− 1
− 3	+ 4	+ 6	0	+ 6	− 1	− 5	− 1
− 3	+ 7	− 13	+ 7	0	− 2	− 8	0
− 5	− 2	+ 5	+ 1	+ 5	+ 4	− 3	− 3

Recall fact 9. *To multiply two signed numbers: Prefix the positive sign to the product of numbers having like signs and the negative sign to the product of numbers having unlike signs.*

EXERCISES

Find the products:

1. $(+7) \cdot (-2)$.
2. $(+4) \cdot (+3)$.
3. $(-2) \cdot (-3)$.
4. $(-2) \cdot (+4)$.
5. $(+7) \cdot (-6)$.
6. $(-8) \cdot (+3)$.
7. $(+3) \cdot (+2) \cdot (+4)$.
8. $(+2) \cdot (-3) \cdot (+4)$.
9. $(-3) \cdot (-3) \cdot (-2)$.
10. $(-4) \cdot (+2) \cdot (-3)$.

11. What is the effect upon a number of multiplying it by $+1$? by -1?

Recall fact 10. *To divide two signed numbers: Prefix the positive sign to the quotient of numbers having like signs, and the negative sign to the quotient of numbers having unlike signs.*

EXERCISES

State the missing number:

1. $(+24) \div (+3) = ?$
2. $(-16) \div (-8) = ?$
3. $(+4) \div (-2) = ?$
4. $(-4) \div (+2) = ?$
5. $(-6) \div (-3) = ?$
6. $(-12) \div ? = (-2)$.

7. What is the effect upon a number if it is divided by $+1$? -1?

RECALL TEST

1. What is a term? Write an algebraic expression of three terms.

2. What is a coefficient? a numerical coefficient? a literal coefficient?

3. What is the coefficient of x in $3ax$? in abx? in $2(a+b)x$?

4. What is the numerical coefficient in $3ab$? $2ab$? ab?

5. What is the literal coefficient of y in $2aby$? in mxy? in $(a+b)y$?

6. What is an exponent? Write a term containing 3 as an exponent; 2 as an exponent.

7. Write a term containing 3 as an exponent and 2 as a numerical coefficient.

8. What is the exponent of b in the term $3\,ab^2$? of a?

9. Give an example of a monomial; a binomial; a trinomial; a polynomial.

10. What are similar terms? Write two terms which are similar; two terms which are not similar.

11. What name is given to each of the 2's in $a^2 - 2\,a$?

12. What is meant by a rational integral polynomial?

Recall fact 11. *Order of fundamental processes in an algebraic expression.*

1. *All processes within a parenthesis are performed first.*
2. *Perform the multiplications, then the divisions in order from left to right.*
3. *Perform the additions and subtractions in order from left to right.*

Recall fact 12. *To add polynomials, arrange similar terms under similar terms, and add each column of similar terms separately.*

Illustrative example. Add $2\,x^3 - 3\,x^2 + 6\,x + 5$, $x^2 - 9\,x - x^3$, $2\,x^3 - 8 + 3\,x^2$, and $7 + 5\,x^2 - 3\,x$.

Solution | *Check*

Let $x = 2$.

$$
\begin{array}{rcrcr}
2\,x^3 - 3\,x^2 + 6\,x + 5 & = & 16 - 12 + 12 + 5 & = & +\,21 \\
-\,x^3 + x^2 - 9\,x \phantom{{}+5} & = & -\,8 + 4 - 18 \phantom{{}+5} & = & -\,22 \\
2\,x^3 + 3\,x^2 \phantom{{}-6\,x} - 8 & = & 16 + 12 - 8 \phantom{{}+5} & = & +\,20 \\
+\,5\,x^2 - 3\,x + 7 & = & 20 - 6 + 7 \phantom{{}+5} & = & +\,21 \\
\hline
3\,x^3 + 6\,x^2 - 6\,x + 4 & = & 24 + 24 - 12 + 4 & = & +\,40 \\
 & & +\,40 & = & +\,40.
\end{array}
$$

EXERCISES

Find the algebraic sum of each of the following:

1. $\begin{array}{l} 5\,D \\ \underline{3\,D} \end{array}$

2. $\begin{array}{l} 5 \sin x \\ \underline{3 \sin x} \end{array}$

3. $\begin{array}{l} 2\,a^2b \\ \underline{3\,a^2b} \end{array}$

4. $\begin{array}{l} -\,5\,mn \\ \underline{-\,3\,mn} \end{array}$

5. $\begin{array}{l} 2 \cos x \\ \underline{3 \cos x} \end{array}$

6. $\begin{array}{l} -\,5\,m^2n \\ \underline{-\,2\,m^2n} \end{array}$

7. $\begin{array}{l} -\,7\,a^2bc \\ \underline{+\,8\,a^2bc} \end{array}$

8. $\begin{array}{r} 4 \tan x \\ \underline{-\,3 \tan x} \end{array}$

9. $\begin{array}{l} +\,9\,abc^2 \\ \underline{-\,3\,abc^2} \end{array}$

10. $\begin{array}{l} -\,4 \sin x \\ \underline{-\quad \sin x} \end{array}$

11. $\begin{array}{l} -\quad \tan x \\ \underline{+\,5 \tan x} \end{array}$

12. $\begin{array}{r} 15 \cos x \\ \underline{-\,13 \cos x} \end{array}$

13. $\begin{array}{r} (a+b)m \\ \underline{-\,2\,bm} \end{array}$

14. $\begin{array}{l} x(a+b) \\ \underline{y(a+b)} \end{array}$

15. $\begin{array}{l} -\,5\,mn \\ \underline{+\;\;amn} \end{array}$

Add and check your results:

16. $3\,a^3 - 2\,a^2 + 3\,a - 5,\ 2\,a^3 - 7\,a + 12,\ 4\,a^2 - a^3 - 3.$

17. $4\,x^3 + 10\,x^2 - 16\,x + 8,\ 2\,x - 12 - 15\,x^2 - 5\,x^3.$

18. $3 \tan x + 5 \cos x - 7 \sin x,\ \ 5 \tan x - 2 \cos x + 3 \sin x,$ $2 \cos x - 2 \tan x,\ 5 \sin x + 6 \cos x - 4 \tan x.$

Recall fact 13. *To subtract one polynomial from another, arrange similar terms under similar terms, and subtract each set of similar terms separately.*

Illustrative example. From $3\,x^3 + 5\,x^2 - 8\,x - 7$ subtract $x^3 + 5\,x - 3\,x^2 - 9$.

Solution *Check*

Let $x = 2$.

$$\begin{array}{rcr} 3\,x^3 + 5\,x^2 - \ \ 8\,x - 7 & = & 24 + 20 - 16 - \ \ 7 = +\,21 \\ \underline{x^3 - 3\,x^2 + \ \ 5\,x - 9} & = & \underline{8 - 12 + 10 - \ \ 9 = -\ \ 3} \\ 2\,x^3 + 8\,x^2 - 13\,x + 2 & = & 16 + 32 - 26 + \ \ 2 = +\,24 \\ & & +\,24 = +\,24. \end{array}$$

You can also check subtraction examples by addition.

EXERCISES

Subtract:

1. $\begin{array}{r} -3M \\ \underline{+2M} \end{array}$ **4.** $\begin{array}{r} -x^2y^2 \\ \underline{-x^2y^2} \end{array}$ **7.** $\begin{array}{r} -3\sin A \\ \underline{-5\sin A} \end{array}$

2. $\begin{array}{r} +2a_1a_2 \\ \underline{+8a_1a_2} \end{array}$ **5.** $\begin{array}{r} +4\cos x \\ \underline{-2\cos x} \end{array}$ **8.** $\begin{array}{r} \cos x \\ \underline{-5\cos x} \end{array}$

3. $\begin{array}{r} -9ab^3 \\ \underline{16ab^3} \end{array}$ **6.** $\begin{array}{r} -7\tan x \\ \underline{+2\tan x} \end{array}$ **9.** $\begin{array}{r} -2\tan x \\ \underline{\tan x} \end{array}$

10. From $2x^3 - 7x^2 + 5x + 3$ subtract $x^3 - x^2 + 8x - 4$.

11. Subtract $5x - 8 + 3x^3$ from $5x^3 - x^2 + 3x + 5$.

12. From $x + y$ subtract $a - b$.

13. What must be added to $m^3 - 7mn - n^2$ to give zero?

14. By how much does $3p^2 - 7pq - q^2$ exceed $2p^2 - 8pq - 3q^2$?

15. From $2\tan x - 3\cos x + 5\sin x$ subtract $\tan x + 2\cos x - 3\sin x$.

16. From $\tan^3 x + 5\tan^2 x - 7$ subtract $\tan^3 x - 3\tan^2 x + 3$.

17. Subtract $5\sin^2 x - 7\tan^5 x - 2\sin x \tan x$ from $3\sin^2 x + 5 - 3\sin x \tan x + 2\tan^5 x$.

18. What must be added to $\sin x + 3\cos x - \tan x$ to give 0?

19. By how much does $3\sin x \cos x - 4\tan^2 x - \cos^2 x$ exceed $\sin x \cos x + 2\tan^2 x + \cos^2 x$?

20. From the sum of $2x^2 + 3x - 5$ and $5x^2 - 7x + 3$ subtract $3x^2 + 4x + 8$.

21. From the difference between $3x^3 + 4x^2 - 7x + 8$ and $2x^3 + 15x - 4$ subtract $x^3 - 8x^2 + 7x - 3$.

Recall fact 14. *The exponent of each quantity in a product equals the sum of the exponents of that quantity in the factors.*

Recall fact 15. *To multiply two monomials, find the product of their numerical coefficients and prefix this product to the product of the literal factors.*

Illustrative examples.

Example 1. Find the product of $+ 2a^2b^3$ and $- 3a^3b^4$.

$$(+2a^2b^3)(-3a^3b^4) = -6a^5b^7.$$

Example 2. Find the product of $\tan x$ and $\tan x$.

$$(\tan x)(\tan x) = (\tan x)^2 = \tan^2 x.$$

Something to think about. How are three or more monomials multiplied?

EXERCISES

Find the products:

1. $(a^4)(-2a^5)$. **4.** $(-x^3y^2)(4x^6y)$. **7.** $(2^3)(2^4)$.

2. $(-2ab^3)(-4b^2)$. **5.** $(-2a^2)(-4b^3)$. **8.** $(7)(7^2)$.

3. $(-a^3)(-a^2b^4)$. **6.** $(3m^2)(a^4)$. **9.** $(10)(10^2)(10^4)$.

Multiply:

10. $\sin x$ by $\sin x$. **13.** $\tan A \cos B$ by $\tan A \cos B$.

11. $\cos x$ by $\cos x$. **14.** $\sin^3 x \cos x$ by $\sin x \cos x$.

12. $\sin x$ by $\sin^2 x \cos x$. **15.** $\sec x \csc^2 x$ by $\sec^2 x \csc x$.

Recall fact 16. *To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial.*

Illustrative example. Multiply $2x^2 - 5x - 7$ by $3x$.

$$3x(2x^2 - 5x - 7) = 6x^3 - 15x^2 - 21x.$$

EXERCISES

Perform the indicated multiplications:

1. $x(a + b - c)$. **4.** $-3x(2x^2 - 3x - 2)$.

2. $-1(2a - 3b + c)$. **5.** $x^2y(-3x + 2y - 3xy)$.

3. $-(4m^2 + 2mn - n^2)$. **6.** $4a^2b^3(a^2 - 2ab - 3b^2)$.

7. $\tan x\,(\tan x - \tan^2 x + \sin^2 x)$.

8. $\cos x\,(\cos^2 x - \tan x - \cos^5 x)$.

9. $\tan A \cos B\,(\tan A - \cos B + \sin A)$.

Simplify:

10. $6x - 2(x - 1)$.

11. $4a + 3(a - 5)$.

12. $6(a - b) - 3(2a + b)$.

13. $-3(L_1 - 3L_2) + 4(L_1 - 5L_2)$.

14. $(\tan x + 1) - 2(\tan x + 3)$.

Recall fact 17. *To multiply a polynomial by a polynomial:*

1. *Arrange both polynomials in descending powers of the same letter.*

2. *Multiply the multiplicand by each term of the multiplier.*

3. *Arrange similar terms under similar terms.*

4. *Add the partial products.*

Illustrative example. Multiply $a^2 - 2ab - b^2$ by $2a - b$.

Solution *Check*

Let $a = 3$ and $b = 2$.

$$
\begin{array}{rrrrcr}
a^2 & -\,2ab & -\,b^2 & & = & 9 - 12 - 4 = -7 \\
2a & -\,b & & & = & 6 - 2 = 4 \\
\hline
2a^3 & -\,4a^2b & -\,2ab^2 & & = & -28 \\
 & -\,a^2b & +\,2ab^2 & +\,b^3 & & \\
\hline
2a^3 & -\,5a^2b & & +\,b^3 & = & 54 - 90 + 8 = -28.
\end{array}
$$

EXERCISES

Perform the indicated multiplications and check your results:

1. $(x - 1)(x^2 - 2x - 3)$.

2. $(x - y)(x^2 - 5xy - y^2)$.

3. $(2a - b)(a^2 - 5ab - b^2)$.

4. $(2a^2 + 3ab - b^2)(3a^2 - ab + 2b^2)$.

5. $(\sin A + \cos B)(\tan A + \cos B)$.

6. $(\tan x + \sin x)^2$.

7. $(2\cos x + 3\tan x)^2$.

8. $(a + b)^2$.

9. $(a + b)^3$.

10. $(a + b)^4$.

The binomial theorem. In each of the last three examples in multiplication, you raised the binomial $(a + b)$ to a power. Raising a binomial to any power is called *expanding the binomial.* The product obtained is called an *expansion.* To expand a binomial by multiplication is a long tedious job. Let us see if we can discover a shorter method.

By actual multiplication:

$$(a + b)^2 = a^2 + 2\,ab + b^2.$$
$$(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$
$$(a + b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4.$$

Let us study these expansions and from them try to get the answers to these questions:

1. How many terms are there in the first expansion? the second? the third? The number of terms in each expansion is ? more than the exponent of the corresponding binomial.

2. What is the exponent of the first and last terms of the first expansion? the second? the third? The exponents of the ? term and the ? term of each expansion are the same as the exponent of the corresponding binomial.

3. Reading the terms from left to right, does the exponent of a increase or decrease? the exponent of b? The exponent of a ? by ? in each term after the first; the exponent of b ? by ? in each term after the second.

4. What is the sum of the exponents of a and b in each term of the first expansion? the second? the third? The sum of the exponents of a and b in each term is the same as the ? of the corresponding binomial.

5. What is the coefficient of the first and last terms of the first expansion? the second? the third?

6. What is the coefficient of the second and the next to the last terms of the second expansion? the third? The coefficient of the second and the next to the last terms of each expansion is ? .

Let us apply these discoveries to the expansion of $(a + b)^5$. How many terms will the expansion contain? What is the first term? the last term? What will be the exponents of a and b in the second term? the third term? Each of the other terms except the last? What will be the coefficient of the second term? the next to the last term? Thus, we can write:

$$(a + b)^5 = a^5 + 5\,a^4b + \ ?\ \ a^3b^2 + \ ?\ \ a^2b^3 + 5\,ab^4 + b^5.$$

There still remains the problem of finding the coefficients of the third and fourth terms. To enable us to see how these coefficients are obtained, let us write the three original expansions in a different form, thus:

$$(a + b)^2 = a^2 + \frac{2}{1}a^1b^1 + \frac{2 \times 1}{1 \times 2}b^2.$$

$$(a + b)^3 = a^3 + \frac{3}{1}a^2b^1 + \frac{3 \times 2}{1 \times 2}a^1b^2 + \frac{3 \times 2 \times 1}{1 \times 2 \times 3}b^3.$$

$$(a + b)^4 = a^4 + \frac{4}{1}a^3b^1 + \frac{4 \times 3}{1 \times 2}a^2b^2 + \frac{4 \times 3 \times 2}{1 \times 2 \times 3}a^1b^3.$$

$$+ \frac{4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4}b^4.$$

7. The number of factors in the numerator is the ? as the number of factors in the denominator.

8. The first factor of each coefficient is the ? as the exponent of the corresponding binomial.

9. The number of factors in the numerator of a coefficient is the same as the exponent of ? in that term.

Now let us find the coefficients of the two middle terms in the expansion of $(a + b)^5$. The coefficient of the third term will have how many factors in the numerator? in the denominator? What will be the first factor in the numerator? The coefficient, then, will be $\frac{5 \times 4}{1 \times 2}$. What will be the coefficient of the fourth term?

In general, then, to expand a binomial to the nth power, the following formula may be used:

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1 \times 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}a^{n-3}b^3 + \cdots.$$

This formula expresses in symbols the law known as the *binomial theorem*, by which we may expand any binomial to an indicated power.

Illustrative examples.

Example 1. Expand $(x+2)^4$.

Solution

In this example $a = x$, $b = 2$, and $n = 4$.

Substituting these values in the formula:

$$\begin{aligned}(x+2)^4 &= x^4 + \frac{4}{1}x^3(2) + \frac{4 \times 3}{1 \times 2}x^2(2)^2 + \frac{4 \times 3 \times 2}{1 \times 2 \times 3}x(2)^3 \\ &\quad + \frac{4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4}(2)^4 \\ &= x^4 + 8x^3 + 24x^2 + 32x + 16.\end{aligned}$$

Example 2. Expand $(2x-3)^5$.

Solution

In this example $a = 2x$, $b = -3$, and $n = 5$.

$$\begin{aligned}(2x-3)^5 &= (2x)^5 + \frac{5}{1}(2x)^4(-3) + \frac{5 \times 4}{1 \times 2}(2x)^3(-3)^2 \\ &\quad + \frac{5 \times 4 \times 3}{1 \times 2 \times 3}(2x)^2(-3)^3 \\ &\quad + \frac{5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4}(2x)(-3)^4 \\ &\quad + \frac{5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5}(-3)^5 \\ &= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243.\end{aligned}$$

EXERCISES

Group A

Expand each of the following:

1. $(a+b)^6$.
2. $(x-2)^4$.
3. $(2x+1)^5$.
4. $(2-b)^4$.
5. $(a+2b)^5$.
6. $(2x-3)^4$.
7. $(a-4b)^3$.
8. $(2x+3y)^4$.
9. $(3x-2y)^5$.

Group B

Expand:

10. $\left(1+\frac{x}{4}\right)^3$.
11. $\left(x+\frac{3}{y}\right)^4$.
12. $\left(\frac{x}{2}-2\right)^5$.
13. $\left(x-\frac{2}{y}\right)^4$.
14. $(x^2-y)^6$.
15. $(1+.03)^3$.

Find the first four terms of the expansion of:

16. $(ax^2+b)^{10}$.
17. $\left(\frac{2}{x}-y\right)^8$.

Group C

Expand:

18. $\left(cx+\frac{2y}{x}\right)^5$.
19. $\left(\frac{x}{2}-\frac{y}{3}\right)^4$.
20. $\left(\frac{m}{n}+\frac{n}{m}\right)^3$.
21. $\left(\frac{x}{c}+c^2\right)^6$.
22. $(\tan x+1)^3$.
23. $(2-\cos x)^5$.
24. $(\sin^2 x-1)^3$.
25. $(\sin x+\cos x)^3$.

26. Find the first three terms of the expansion of $(m+n)^a$.

27. You will learn later that the amount of \$1 at 4% interest, compounded quarterly, for $1\frac{1}{2}$ years, is $(1+.01)^6$. Find the value of the expansion correct to 6 decimal places.

28. Expand: (*a*) $(1+.04)^5$; (*b*) $(1.01)^5$; (*c*) $(1.03)^4$.

Finding any term in a binomial expansion. Let us see if we can build a formula for any term of an expansion of a binomial without writing all the terms that precede it.

Consider the fourth, fifth, and sixth terms of the expansion of $(a + b)^n$:

$$\cdots \frac{n(n-1)(n-2)}{1 \times 2 \times 3} a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4} a^{n-4}b^4 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \times 2 \times 3 \times 4 \times 5} a^{n-5}b^5 \cdots$$

1. The denominator of the coefficient of the fourth term is $1 \times 2 \times 3$. What is the denominator of the coefficient of the fifth term? the sixth? Notice that in each case the last factor in the denominator is one less than the number of the term. In general, then, the denominator of the rth term would be $1 \times 2 \times 3 \cdots \times (r - 1)$.

2. The numerator of the coefficient of the fourth term is $n(n - 1)(n - 2)$. What is the numerator of the coefficient of the fifth term? the sixth? Notice that the last factor is n minus two less than the number of the term. We could write the last factor of the numerator of the coefficient of the rth term, then, as $n - (r - 2)$, which is the same as $n - r + 2$. The numerator of the coefficient of the rth term, then, is $n(n - 1)(n - 2) \cdots (n - r + 2)$.

3. The exponent of a in the fourth term is $n - 3$. What is the exponent of a in the fifth term? the sixth term? Notice that in each case the exponent of a is n minus one less than the number of the term. The exponent of a in the rth term, then, is $n - (r - 1)$, which is the same as $n - r + 1$.

4. The exponent of b in the fourth term is 3. What is the exponent of b in the fifth term? the sixth term? Notice that in each case the exponent of b is one less than the number of the term. Hence, the exponent of b in the rth term is $r - 1$.

Putting these conclusions together, the rth term is:

$$\frac{n(n-1)(n-2)\cdots(n-r+2)}{1\times 2\times 3\cdots(r-1)}a^{n-r+1}b^{r-1}.$$

Illustrative example. Find the fourth term of $(x+3y)^7$.

Solution

In this example $r=4$, $a=x$, $b=3y$, $n=7$.

The fourth term, then, $=\dfrac{7\times 6\times 5}{1\times 2\times 3}x^{7-4+1}(3y)^{4-1}$

$= 35x^4(27y^3)$

$= 945x^4y^3$.

EXERCISES

Group A

Find the:

1. 3rd term of $(x+y)^5$.
2. 5th term of $(x-y)^7$.
3. 4th term of $(a+2)^4$.
4. 5th term of $(x+2)^8$.
5. 11th term of $(2x+b)^{12}$.
6. 3rd term of $(3x-1)^9$.
7. 5th term of $(m-2n)^8$.
8. 4th term of $(3x-2y)^8$.

Group B

9. 5th term of $(m^2-n)^8$.
10. 6th term of $(m+\frac{1}{2}n)^7$.
11. 5th term of $(a^2-2b^2)^8$.
12. 20th term of $(1-x)^{25}$.

Group C

13. 4th term of $\left(x+\dfrac{1}{x}\right)^9$.
14. 5th term of $\left(\dfrac{2}{a}+\dfrac{a}{2}\right)^8$.
15. 6th term of $\left(3x+\dfrac{4}{x}\right)^9$.
16. middle term of $(m-n)^8$.
17. middle term of $\left(x-\dfrac{1}{x}\right)^{10}$.
18. middle term of $\left(\dfrac{3}{a}-\dfrac{a}{3}\right)^{12}$.

Recall fact 18. *The exponent of each quantity in a quotient equals the exponent of that quantity in the dividend less the exponent of the same quantity in the divisor.*

Recall fact 19. *To divide two monomials, find the quotient of their numerical coefficients and prefix this quotient to the quotient of the literal factors.*

Illustrative examples.

Example 1. $-10\,a^5b^2 \div 2\,a^3b = -5\,a^2b.$

Example 2. $\tan^5 x \div \tan^2 x = \tan^3 x.$

EXERCISES

Perform the indicated divisions:

1. $x^8 \div x^3.$
2. $2^5 \div 2^2.$
3. $x^8 \div x^8.$
4. $12\,a^4 \div 3\,a^2.$
5. $K_1{}^5K_2{}^3 \div K_1{}^2K_2{}^2.$
6. $3\,a^2b^3 \div ab^2.$
7. $\frac{1}{3}\pi r^2h \div \pi r^2.$
8. $\frac{4}{3}\pi r^3 \div \frac{1}{3}\pi rh.$
9. $7\,a^5bx \div 3\,ab^2x.$
10. $\tan^5 x \div \tan x.$
11. $\sin^4 x \div \sin^3 x.$
12. $15\tan^6 x \div 3\tan^2 x.$
13. $\tan^2 x \div \tan^2 x.$
14. $4\cos^3 x \div \cos^3 x.$
15. $5\sin x\cos x \div \sin x.$
16. $3\sin^3 x \div \sin^2 x.$
17. $\tan^2 x\cot x \div \cot x.$
18. $\sec^3 x \div \csc^2 x\sec x.$

Recall fact 20. *To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.*

Illustrative examples.

Example 1. $(8\,a^3b^4 - 4\,a^2b^2 + 2\,a^3bc^2 - 2\,ab) \div 2\,ab$
$$= 4\,a^2b^3 - 2\,ab + a^2c^2 - 1.$$

Example 2. $(3\tan^2 x - 9\tan x\cos x + 18\tan^5 x) \div 3\tan x$
$$= \tan x - 3\cos x + 6\tan^4 x.$$

EXERCISES

Divide:

1. $a^3 - 2a^2 + a^4$ by a^2.

2. $5a^2y - 10a^2y^2 + 15ay^3$ by $5ay$.

3. $9A^2B^2 + 18AB^2 - 27A^3B^4$ by $-9AB^2$.

4. $\sin^2 x - \sin^3 x$ by $\sin^2 x$.

5. $3 \tan x - \tan^2 x$ by $\tan x$.

6. $9 \sin^2 A - 18 \sin^3 A$ by $9 \sin^2 A$.

Recall fact 21. *To divide a polynomial by a polynomial:*

1. *Arrange the divisor and dividend in ascending or descending powers of the same letter.*

2. *Divide the first term of the dividend by the first term of the divisor and place this result as the first term of the quotient.*

3. *Multiply the whole divisor by this first term of the quotient and subtract the result from the dividend.*

4. *If the division is not yet complete, bring down the next term (or terms) of the dividend and repeat the process.*

Illustrative example. Divide $2x^4 - 5x^3 + 2x^2 + 13x - 12$ by $x^2 - 3x + 4$.

Solution

$$
\begin{array}{r r}
 & 2x^2 + x - 3 \\
x^2-3x+4\,) & 2x^4-5x^3+2x^2+13x-12 \\
 & \underline{2x^4-6x^3+8x^2} \qquad\qquad \\
 & x^3-6x^2+13x \qquad \\
 & \underline{x^3-3x^2+\ \ 4x} \qquad \\
 & -3x^2+\ \ 9x-12 \\
 & \underline{-3x^2+\ \ 9x-12}
\end{array}
$$

Check

Let $x = 2$.

Dividend $= +14$.

Divisor $= +2$.

Quotient $= +7$.

$(+2) \times (+7) = (+14)$.

$14 = 14$.

Multiply the divisor by the quotient. Does it equal the dividend?

EXERCISES

Divide:

1. $6a^2 + 17a + 12$ by $2a + 3$.
2. $12a^2 - 5ab - 2b^2$ by $3a - 2b$.
3. $x^3 + 3x^2 + 3x + 1$ by $x + 1$.
4. $6a^3 - a^2 - 26a + 21$ by $2a - 3$.
5. $4x^4 - 13x^3 + 40x^2 - 12x + 9$ by $x^2 - 3x + 9$.
6. $\tan^2 x - 13 \tan x + 12$ by $\tan x - 1$.
7. $\sin^2 x - 3 \sin x - 10$ by $\sin x + 2$.
8. $6 \tan^3 x + 13 \tan^2 x + 8 \tan x + 3$ by $2 \tan x + 3$.
9. $6 \cos^3 x - 17 \cos^2 x \tan x + 2 \cos x \tan^2 x + 15 \tan^3 x$ by $3 \cos^2 x - 4 \cos x \tan x - 5 \tan^2 x$.
10. $a^4 + a^2b^2 + b^4$ by $a^2 + ab + b^2$.
11. $a^4 - b^4$ by $a^2 + b^2$.
12. $a^3 - b^3$ by $a - b$.
13. $x^6 - y^6$ by $x^4 + x^2y^2 + y^4$.
14. $a^3 + b^3$ by $a^2 - ab + b^2$.
15. The area of a rectangle is $14x^2 + 43x + 20$. Find its width and perimeter if its length is $7x + 4$.

How Old Is Algebra

The oldest known mathematical manuscript was written by an Egyptian, Ahmes (1650 B.C.) and is commonly known as the "Ahmes Papyrus." Algebra may be said to have had its origin in Egypt because the work of Ahmes contains much that we now regard as algebra. However, the progress of the development of algebra was probably hindered by the lack of suitable symbols, and it was not until mathematicians adopted a unified symbolism for expressing universally accepted concepts that the treatment of the subject reached its present importance.

The earliest printed book in which the + and − signs are found was published in 1489 by John Widmann. Until about

1600 the word *equals* was usually written out in full. The present sign, =, is due to Robert Recorde (1510–1558), the author of *The Whetstone of Witte*, which is the first English treatise on Algebra. He selected this sign because "noe 2 thynges can be moare equalle" than two such parallel lines. The multiplication sign, ×, is probably due to Oughtred (1574–1660); the sign for division, ÷, was first used by Rhan in 1659.

CUMULATIVE REVIEW

Chapters I and II

1. Which of these statements are true? Which are false?

(*a*) If $A = 2B$, $\sin A = 2 \sin B$.

(*b*) As an angle decreases from 90° to 0°, the cotangent of the angle increases more rapidly than the cosine of the angle.

(*c*) If $\cos A = .2419$, $\sin A = .9703$.

(*d*) The sine of an angle can never be greater than unity.

(*e*) If a, b, c are the sides of a right triangle (c is the hypotenuse) and $a < b < c$, then angle A is less than 45°.

(*f*) A binomial is unchanged in value if each of its terms is multiplied by -1.

(*g*) In the expression $3a^3$, the coefficient and the exponent of a are numerically equal.

(*h*) The sum of $a - b$ and $-a + b$ is 1.

(*i*) a exceeds b by $a - b$.

(*j*) x^3 stands for $x \cdot x \cdot x$ while $3x$ stands for $x + x + x$.

2. Complete each of the following statements:

(*a*) If the tangent of an angle is less than 1, the angle is less than ? degrees.

(*b*) The sides of a right triangle are 27, 36, 45. The angle opposite side 36 has ? degrees (correct to the nearest degree).

(*c*) If the acute angles of a right triangle are in the ratio of 2 : 1, the sines of these angles are in the ratio of ? : ? .

(*d*) $\cot 41° = \tan$? °.

(*e*) The reciprocal of the sine is the ? , while the reciprocal of the cosecant is the ? .

(*f*) If the remainder is 32 and the subtrahend is 11, the minuend is ? .

(*g*) The fifth term of the expansion of $\left(x - \frac{2}{x}\right)^7$ is ? .

(*h*) If the difference between two numbers is zero, their quotient is ? .

(*i*) If $6x^2 - 5x - 6$ is exactly divisible by $2x - 3$, it is also exactly divisible by the binomial ? .

(*j*) If $x - 2y$ is the divisor and $x^2 + 2xy + 4y^2$ is the quotient, the dividend is ? .

3. Find the numerical value of: $\tan 30° \cot 30° - 2 \sin^2 60° + \frac{1}{2} \cos^2 45 - \cos 90° \sin 0° + \csc 30°$.

4. If $\cos A = \frac{5}{13}$, construct angle A and write the values of the other five functions of A.

5. If $\tan B = \frac{12}{5}$, construct angle B and write the values of the six functions of the complement of angle B.

6. The two legs of a right triangle are 28 inches and 45 inches respectively. Find the ratio of the sines of the acute angles of the triangle.

7. A street is 80 feet wide. From the roof of a building on one side of the street, the angle of depression of the other side of the street is 73°. Find the height of the building correct to the nearest foot.

8. From the sum of $3x^2 - 2xy - y^2$ and $x^2 + 2xy + 4y^2$ take $4x^2 + 3y^2$.

9. What must be added to the sum of $3m^2 - 2mn + n^2$ and $-m^2 + 2mn - 2n^2$ to produce zero as an answer?

10. By how much does the sum of $\tan^2 x - 2 \tan x - 7$ and $\tan^2 x - 1$ exceed $\tan^2 x + 1$?

11. Divide $6a^3 - 13a^2 + 14a - 12$ by $2a - 3$. Check your work, using $a = 3$.

12. Using the binomial theorem, expand $(3x - 2)^4$.

CHAPTER III. FACTORING

A science is exact only in so far as it employs mathematics. — KANT.

You will recall that the factors of a number or expression are those numbers or quantities which when multiplied together produce that original number or expression.

Recall fact 22. *To find the factors of a polynomial whose terms have a common monomial factor:* (1) *Find the largest common factor of all the terms of the polynomial, and* (2) *divide the polynomial by this common factor to find the other factor.*

Illustrative examples.

Example 1. Factor $2x^2 - 6x + 10xy$.

$$2x^2 - 6x + 10xy = 2x(x - 3 + 5y).$$

Example 2. Factor $\tan^2 x - 3\tan x + 4\tan x \cos x$.

$$\tan^2 x - 3\tan x + 4\tan x\cos x$$
$$= \tan x(\tan x - 3 + 4\cos x).$$

Something to think about. Suggest two ways in which such examples may be checked.

EXERCISES

Factor and check:

1. $.3x^2 - .9x$. **3.** $.2lh + .2lw$. **5.** $\tan^2 x - \tan x$.

2. $\frac{1}{2}hb_1 + \frac{1}{2}hb_2$. **4.** $2\pi rh + 2\pi r^2$. **6.** $\cos^2 x - \cos^3 x$.

7. $a(x - y) - b(x - y)$. **9.** $\tan x \cos x - \tan x \sin x$.

8. $(x - y)^2 - (x - y)$. **10.** $3.1416 \times 7.324 + 3.1416 \times 2.676$.

Recall fact 23. *The factors of the difference of two squares are the sum and difference of their square roots.*

Illustrative examples.

Example 1. Factor $a^2 - b^2$.

$$a^2 - b^2 = (a + b)(a - b).$$

Example 2. Factor $\tan^2 x - \cos^2 x$.

$$\tan^2 x - \cos^2 x = (\tan x + \cos x)(\tan x - \cos x).$$

EXERCISES

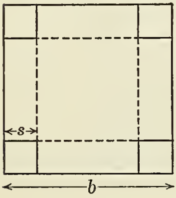
Factor each of the following and check:

1. $x^2 - 4$.
2. $.04\,a^2 - 9\,b^2$.
3. $1.21 - t^2$.
4. $a^2b^2 - 36\,c^2d^2$.
5. $7^2 - 2^2$.
6. $113^2 - 87^2$.
7. $a^3 - a$.
8. $2\,m^2 - 8$.
9. $128 - 2\,x^2$.
10. $\frac{1}{2}\,mv_1^2 - \frac{1}{2}\,mv_2^2$.
11. $\pi R^2 - \pi r^2$.
12. $\sin^2 x - \cos^2 x$.
13. $4\cos^2 x - \tan^2 x$.
14. $16\tan^4 x - 4\cos^4 x$.
15. $9\cos^3 x - \cos x$.
16. $(a + b)^2 - c^2$.
17. $(a - 2\,b)^2 - 9$.
18. $x^2 - (y - z)^2$.

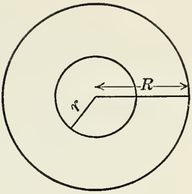
19. Write these numbers as the sum and difference of two numbers and find their products: 105×95; 102×98; 110×90; 32×28.

20. Find the value of: $324^2 - 323^2$; $765^2 - 764^2$; $536^2 - 464^2$; $867^2 - 857^2$.

21. Open square tin boxes are frequently made by cutting out equal squares from the corners of a square piece of tin and folding along the dotted lines, as shown in the figure. (*a*) Make a formula for the area of the tin in the box; and (*b*) Factor your formula and use it to find the area used to make a box such that $b = 10$ in. and $s = 1\frac{1}{2}$ in.



22. Make a formula for the area of the ring made by cutting a circle of radius r from a circle of radius R.



23. Four circular holes (radius r) were cut into a circular disc of radius R. Show that the area of the tin remaining is $\pi(R + 2r)(R - 2r)$, and find this area if $R = 12$ in. and $r = 2$ in.

Recall fact 24. *The binomial factors of a quadratic trinomial must satisfy these three conditions:* (1) *the first terms must be factors of the first term of the trinomial,* (2) *the last terms must be factors of the last term of the trinomial, and* (3) *these factors must be such that when the product of the outer terms of the binomials is added to the product of the inner terms the sum is the middle term of the trinomial.*

Illustrative examples.

Example 1. Factor $x^2 - 5x - 6$.

$$x^2 - 5x - 6 = (x - 6)(x + 1).$$

Example 2. Factor $2a^2 - ab - 6b^2$.

$$2a^2 - ab - 6b^2 = (2a + 3b)(a - 2b).$$

Example 3. Factor $\sin^2 x - 3\sin x + 2$.

$$\sin^2 x - 3\sin x + 2 = (\sin x - 1)(\sin x - 2).$$

Something to think about. Suggest three ways in which such examples may be checked.

EXERCISES

Factor each of the following and check:

1. $a^2 - 7a + 12$.
2. $y^2 + .2y - .15$.
3. $y^2 - y - 12$.
4. $a^2 - 1.1ab + .30b^2$.
5. $S^2 + 2S - 80$.
6. $R^2 - RS + .16S^2$.

7. $-t^2 + 15 - 2t$.

8. $-15 + 2a^2 - 7a$.

9. $-7xy + 4y^2 + 3x^2$.

10. $9x^2 + 3x - 2$.

11. $10 - 29x + 10x^2$.

12. $\tan^2 x + 6\tan x - 16$.

13. $\sin^2 x + 9\sin x + 14$.

14. $5\tan^2 x + \tan x - 6$.

15. $6\cos^2 x + 13\cos x + 6$.

16. $-10 + 15\sin^2 x + 19\sin x$.

Something to think about.

1. The equal factors of 49 are ? and ? . The number 49 is called a ? . As the factors of $x^2 + 6x + 9$ are ? and ? , the trinomial $x^2 + 6x + 9$ is a ? .

2. Must the first and last terms of a trinomial that is a perfect square be perfect squares?

Supply the missing term so that each of the following trinomials shall be a perfect square:

17. $x^2 - 4x + ?$.

18. $x^2 + ?\,x + 9$.

19. $m^2 + ? + 4n^2$.

20. $? - 4xy + 4y^2$.

21. $x^2 + 12x + ?$.

22. $? - 6x + ?$.

Something to think about.

1. Can the method of factoring a quadratic trinomial be applied to factoring the difference of two squares? Try it.

2. If one of two factors of a positive monomial increases, then its other factor must ? .

3. If one factor of a quantity is known, how may the other be found?

4. Can factoring examples be checked by substitution?

What do we know about factors whose product is zero? Complete the following:

1. $8 \times 0 = ?$

2. $0 \times (-3) = ?$

3. $(x + 2) \times 0 = ?$

4. $0 \times -(x - 4) = ?$

5. What is the value of $(x - 2)(x - 3)$ if $x = 3$? if $x = 2$?

6. If one factor of an indicated product is zero, the product is ? .

7. If a product is zero, at least one of its factors must be ? .

Remember

If the product of two or more factors is zero, at least one of the factors must equal zero.

Recall fact 25. *A quadratic equation is an equation in which the highest exponent of the unknown quantity is 2.*

Illustrative examples.

Example 1. Solve $2x^2 - 11x = 6$ by factoring.

Solution

$$2x^2 - 11x - 6 = 0.$$
$$(x - 6)(2x + 1) = 0.$$
$$x - 6 = 0. \quad \Big| \quad 2x + 1 = 0.$$
$$x = 6. \quad \Big| \quad x = -\tfrac{1}{2}.$$

Check

Substituting $x = 6$,
$$72 - 66 = 6.$$
$$6 = 6.$$
Substituting $x = -\frac{1}{2}$,
$$\tfrac{1}{2} + \tfrac{11}{2} = 6.$$
$$6 = 6.$$

Example 2. Solve $\sin^2 x - 2\sin x + 1 = 0$ by factoring.

Solution

$$\sin^2 x - 2\sin x + 1 = 0.$$
$$(\sin x - 1)(\sin x - 1) = 0.$$
$$\sin x - 1 = 0. \quad \Big| \quad \sin x - 1 = 0.$$
$$\sin x = 1. \quad \Big| \quad \sin x = 1.$$
$$x = 90°. \quad \Big| \quad x = 90°.$$

Check

Substituting $x = 90°$,
$$\sin^2 90° - 2\sin 90° + 1 = 0.$$
$$(1)^2 - 2(1) + 1 = 0.$$
$$0 = 0.$$

EXERCISES

Solve by factoring and check each answer:

1. $x^2 - 3x + 2 = 0.$
2. $m^2 + 7m + 6 = 0.$
3. $R^2 - 8R = 20.$
4. $f_1^2 = 4f_1 + 77.$
5. $x^2 - 15 = 2x.$
6. $5 - 6y = -y^2.$
7. $-3 + 2x^2 - x = 0.$
8. $-x + x^2 = 0.$
9. $x^2 = -3x.$
10. $x^2 = .09.$
11. $\tan^2 x - 2\tan x + 1 = 0.$
12. $2\sin^2 x = \sin x.$
13. $2\cos^2 x = \cos x.$
14. $1 + 2\cos^2 B = 3\cos B.$

The Factor Theorem

To factor expressions of higher degree than the second, such as $x^3 - 7x - 6$, we must develop a theorem known as the *factor theorem.*

1. Divide $x^3 + x^2 - 5x + 8$ by $x - 3$. What is the quotient? What is the remainder? Find the value of $x^3 + x^2 - 5x + 8$ if $x = 3$. How does this value compare with the remainder you obtained when you divided? Do you think this is always true?

2. Divide $x^3 + 3x^2 - 3x + 2$ by $x - 1$. The remainder is ? . The value of the dividend when $x = 1$ is ? .

3. Divide $x^3 - 7x^2 + 7x + 15$ by $x + 2$. The remainder is ? . The value of the dividend when $x = -2$ is ? .

Thus we observe that the value of the dividend when x equals any number, a, is the same as the remainder obtained by dividing the dividend by $x - a$.

4. Divide $x^3 - 6x^2 + 11x - 6$ by $x - 2$. The remainder is ? .

Since the remainder is 0, then $x - 2$ is an exact divisor of ? , and therefore must be a factor of ? .

If 2 is substituted for x in the dividend, its value is ? , and the remainder therefore is ? . Then $x - 2$ is a ? of $x^3 - 6x^2 + 11x - 6$.

5. The value of $2x^3 - 4x^2 - 3x - 9$ is ? , when 3 is substituted for x. Then $x - 3$ is a ? of $2x^3 - 4x^2 - 3x - 9$.

Thus, in general, if the dividend becomes 0 when a is substituted for x, then ? is a factor of it.

This leads us to a new way of factoring.

Factor Theorem. *If a rational integral polynomial in powers of x has the value 0 when a is substituted for x, then $x - a$ is one of its factors.*

Proof. Let P be the polynomial in x, Q the quotient, and R the remainder when it is divided by $x - a$. Then

$$P = Q(x - a) + R.$$

If, when a is substituted for x, P equals 0, we have

$$0 = Q(a - a) + R$$
$$0 = 0 + R$$
$$\therefore R = 0.$$

Since the remainder is 0, P is exactly divisible by $x - a$, or $x - a$ is a factor of P. Observe also that if $x - a$ is a factor of the polynomial, a must be a factor of the constant term (the term not containing x).

Illustrative examples.

Example 1. Factor $x^3 - 7x - 6$.

Solution: The factors of -6 are $\pm 1, \pm 2, \pm 3, \pm 6$.

Substituting $+1$ for x,

$$x^3 - 7x - 6 = 1 - 7 - 6 = -12 \text{ (not zero).}$$
$$\therefore (x - 1) \text{ is not a factor.}$$

Substituting -1 for x,

$$x^3 - 7x - 6 = -1 + 7 - 6 = 0.$$
$$\therefore [x - (-1)] \text{ or } x + 1 \text{ is a factor.}$$

Dividing $x^3 - 7x - 6$ by $x + 1$ to get the other factors,

$$x^3 - 7x - 6 = (x + 1)(x^2 - x - 6)$$
$$= (x + 1)(x - 3)(x + 2).$$

Example 2. Factor $x^4 - x^3 - 7x^2 + x + 6$.

Solution: The factors of $+6$ are $\pm 1, \pm 2, \pm 3, \pm 6$.

Substituting $+1$ for x,

$$x^4 - x^3 - 7x^2 + x + 6 = 1 - 1 - 7 + 1 + 6 = 0.$$
$$\therefore [x - (+1)] \text{ or } x - 1 \text{ is a factor.}$$

Dividing to get the other factor,

$$x^4 - x^3 - 7x^2 + x + 6 = (x - 1)(x^3 - 7x - 6).$$

Factoring $x^3 - 7x - 6$ by the factor theorem,

$$x^4 - x^3 - 7x^2 + x + 6 = (x - 1)(x + 1)(x - 3)(x + 2).$$

EXAMPLES

Group A

1. Is $x - 1$ a factor of $x^3 - 4x + 3$?
2. Is $x + 1$ a factor of $x^3 + 1$?
3. Show that $x^3 - 3x^2y + 3xy^2 - y^3$ is divisible by $x - y$.

Factor:

4. $x^3 - 7x + 6$.
5. $x^3 + 6x^2 + 11x + 6$.
6. $x^3 - 6x^2 + 11x - 6$.
7. $a^3 + a + 2$.
8. $y^3 + 2y^2 - 9y - 18$.
9. $x^4 - x^3 - 16x^2 + 4x + 48$.

Group B

Factor:

10. $2x^3 + x^2 - 5x + 2$.
11. $3x^3 + 4x^2 - 3x - 4$.
12. $2x^3 + 11x^2 + 19x + 10$.
13. $2x^4 + x^3 - 11x^2 - 4x + 12$.
14. $x^3 + mx^2 - m^2x + 2m^3$.
15. $x^3 - 7a^2x + 6a^3$.

Group C

16. Is $x - 1$ a factor of $x^5 - 1$?
17. Is $x + 1$ a factor of $x^5 - 1$?
18. Is $x + 1$ a factor of $x^5 + 1$?
19. Is $x - 1$ a factor of $x^5 + 1$?
20. Is $10^5 - 1$ divisible by 9 (*i.e.*, $10 - 1$)?
21. Is $10^5 - 1$ divisible by 11?
22. Show by the factor theorem that $a^3 - 1$ is divisible by $a - 1$ but not by $a + 1$.
23. Show by the factor theorem that $a^3 + 1$ is divisible by $a + 1$ but not by $a - 1$.
24. What is the value of k, if $x - 3$ is a factor of $3x^2 - kx + 3$?
25. What is the value of k, if $x + 2$ is a factor of $x^2 - x - k$?

26. Considering the condition, first when n is even and second when n is odd,

(*a*) Is $x - y$ a factor of $x^n + y^n$?
(*b*) Is $x - y$ a factor of $x^n - y^n$?
(*c*) Is $x + y$ a factor of $x^n + y^n$?
(*d*) Is $x + y$ a factor of $x^n - y^n$?

The Origin of the Exponent

The struggle to obtain a simple method of writing the power of a number occupied the attention of mathematicians for centuries and it was not until Vieta (1540–1603) that any forward step was made. Among many epoch-making innovations and standardizations, he employed "A quadratus" to represent x^2 and "A cubus" for x^3, instead of the former cumbersome symbols and devices.

However, the idea of using exponents to indicate the number of times a quantity is used as a factor is credited to Descartes, the brilliant French philosopher and mathematician who lived about 1600.

CUMULATIVE REVIEW

Chapters I, II, and III

1. Which of these statements are true? Which are false?

(*a*) As the tangent of an angle increases, the cosine of the angle decreases.

(*b*) If tan A is greater than 1, then angle A is greater than 45°.

(*c*) A binomial is unchanged in value if each of its terms is divided by -1.

(*d*) In the expression $3x^3y$, the coefficient of x^3 is the same as the exponent of x.

(*e*) If $a - b$ is subtracted from 0, the difference is equal to 0 subtracted from $a - b$.

(*f*) $(2^a - 3^b)$ and $(2^a + 3^b)$ are the factors of $2^{2a} - 3^{2b}$.

(*g*) If $(a - b)$ is a factor of an expression, then $(b - a)$ is also a factor of the same expression.

(*h*) When $y = 90°$, the expression $16 \sin^2 y - 8 \sin y + 1$ is exactly 3 times either one of its factors.

(*i*) $(x + 2)$ is a factor of $x^3 - 3x^2 + x - 5$.

(*j*) The binomial factor of $1 + \cos^3 x$ is $1 + \cos x$.

2. Complete each of the following statements:

(*a*) If $\sin 2x = .9848$, $2x =$? , $x =$? , and $\sin x =$? .

(*b*) The sides of a right triangle are in the ratio of $1 : \sqrt{3} : 2$. The angles of the triangle are in the ratio of ? : ? : ? .

(*c*) If the remainder is x and the subtrahend is y, the minuend is ? .

(*d*) In the expansion of $(x - 3y)^6$, there are ? negative terms and ? positive terms.

(*e*) The middle term of the expansion of $\left(x^2 - \frac{1}{x}\right)^{10}$ is ? .

(*f*) The expression whose factors are 8, $a^2 - b^2$ and $a^2 + b^2$ is ? .

(*g*) With the aid of factoring, the square root of

$$(a^3 - ab^2)(a^2 + ab)(9a - 9b)$$

is readily found to be ? .

(*h*) The monomial factor of $a^{n+1} - a$ is ? .

(*i*) The common factors of $x^3(a - b)^2$ and $x^4(a - b)$ are ? .

(*j*) A binomial factor of $x^3 + 3x^2 + 3x + 9$ is ? .

3. The two legs of a right triangle are 40 inches and 42 inches respectively. Find the ratio of the secants of the acute angles.

4. Two buildings are directly opposite each other on a street 60 feet wide. From the top of the taller building the angles of depression of the top and bottom of the other building are 32° and 48° respectively. Find the height of each building correct to the nearest foot.

5. From $7x^3 + 12x^2 - 5x$ take the sum of $-2x^3 - 5x$ and $4x^3 - 3x^2$.

6. Divide: $15 \tan^3 x + \tan^2 x - 3 \tan x + 2$ by $3 \tan x + 2$. Check your work, using $x = 45°$.

7. Expand $(1.05)^5$, *i.e.*, $(1 + .05)^5$, by the binomial theorem and carry the work of simplification far enough to write the answer correct to the nearest tenth.

8. Find the four prime factors of $x^4y^4 - 10x^2y^2 + 9$.

9. Find the sum of the two factors of $3t^2 + 14t + 15$.

10. (*a*) Find the three factors of: $x^3 - 19x + 30$.

(*b*) For positive values of x, which factor will always have the largest value?

(*c*) For positive values of x, which factor will always have the least value?

11. Solve by factoring and check: $2 \cos^2 x - 3 \cos x + 1 = 0$.

12. Show that $4^a = 2^{2a}$. Factor: $4^a - 9^b$.

CHAPTER IV. FRACTIONS AND IDENTITIES

Business is rapidly becoming a science, but it can do this only in so far as it is capable of using mathematics. — W. W. RANKIN.

FRACTIONS

Measurements of most heights, weights, temperature readings, distances, time, etc., involve fractions. Frequently in our work it becomes necessary to combine several fractions into one fraction. Let us recall the chief facts relating to fractions which we learned in elementary algebra.

Recall fact 26. *Multiplying or dividing both the numerator and the denominator of a fraction by the same expression (not zero) does not change the value of the fraction.*

Do you remember how the signs of a fraction may be changed without changing its value? The following will help you:

1. Since $+\frac{-3}{+4} = +\left(\frac{-3}{+4}\right) = +\left(-\frac{3}{4}\right) = -\frac{3}{4}$, the value of $+\frac{-3}{+4}$ is $-\frac{3}{4}$.

Complete and explain the changes of sign:

2. $+\frac{+3}{+4} = +\left(\frac{+3}{+4}\right) = +\left(\ ?\ \frac{3}{4}\right) = \ ?\ \frac{3}{4}$.

3. $-\frac{+3}{-4} = -\left(\frac{-3}{?}\right) = -\left(\ ?\ \frac{3}{4}\right) = +\frac{3}{4}$.

4. In general $+\frac{-a}{+b} = +\frac{+a}{-b} = -\frac{+a}{+b} = -\frac{-a}{-b}$.

5. In general $+\frac{+a}{+b} = +\frac{-a}{-b} = -\frac{-a}{+b} = -\frac{+a}{-b}$.

Recall fact 27. *Any two of the three signs of a fraction* (i.e., *the sign of the fraction, the sign of the numerator, and the sign of the denominator*) *can be changed without changing the value of the fraction.*

Something to think about. Show how recall fact 27 may be used to transform:

1. $\dfrac{-x}{(y-z)}$ into $\dfrac{x}{(z-y)}$.

2. $\dfrac{-x}{(y-x)(x-z)}$ into $\dfrac{x}{(x-y)(x-z)}$.

Recall fact 28. *To reduce a fraction to its lowest terms, factor both terms of the fraction and divide both terms by their highest common factor.*

Illustrative examples.

Example 1. Reduce $\dfrac{72\,x^3y}{90\,x^2y^3}$ to lowest terms.

Analysis

The largest common factor of the terms of the fraction is $18\,x^2y$. This is called the highest common factor (H. C. F.) of the numerator and the denominator, since it contains all the factors that are common to both numerator and denominator.

Solution	*Check*
Dividing both numerator and denominator by their H. C. F., *i.e.*, $18\,x^2y$, we obtain $\dfrac{4\,x}{5\,y^2}$.	Let $x = 2$ and $y = 3$.
$\therefore \dfrac{72\,x^3y}{90\,x^2y^3} = \dfrac{4\,x}{5\,y^2}$.	$\dfrac{72 \times 8 \times 3}{90 \times 4 \times 27} = \dfrac{4 \times 2}{5 \times 9}$.
	$\frac{8}{45} = \frac{8}{45}$.

Example 2. Reduce $\dfrac{x^2y - xy - 6y}{x^2y^2 - 9y^2}$ to lowest terms.

Solution

$$\frac{x^2y - xy - 6y}{x^2y^2 - 9y^2} = \frac{y(x-3)(x+2)}{y^2(x-3)(x+3)} = \frac{x+2}{y(x+3)}.$$

Check

Let $x = 2$ and $y = 3$.

$$\frac{12 - 6 - 18}{36 - 81} = \frac{2+2}{3(2+3)}$$

$$\tfrac{4}{15} = \tfrac{4}{15}.$$

Something to think about.

1. Does $\dfrac{2a + b}{xb} = \dfrac{2a}{x}$?

2. Does $\dfrac{2a + (x+y)}{(x+y)} = \dfrac{2a}{1}$?

3. Does $\dfrac{2a(x+y)}{(x+y)} = \dfrac{2a}{1}$?

Test your conclusions by substituting numbers for the letters.

EXERCISES

Group A

Reduce each of the following to lowest terms:

1. $\dfrac{3x^9y^7}{6x^7y}$.

2. $\dfrac{-8x^3}{2x^2 - 6x^3}$.

3. $\dfrac{3x^2 + 3xy}{x^3 - xy^2}$.

4. $\dfrac{a-1}{1-a}$.

5. $\dfrac{x^2 + x - 12}{3x^2 + 9x - 54}$.

6. $\dfrac{(x+y)^2 - 1}{x^2z + xyz + xz}$.

7. $\dfrac{\pi R^2 - \pi r^2}{2\pi R + 2\pi r}$.

8. $\dfrac{\frac{1}{2}at_1^2 - \frac{1}{2}at_2^2}{at_1 + at_2}$.

9. $\dfrac{\sin^2 x - 1}{\sin x + 1}$.

10. $\dfrac{2\cos^2 x - 2\sin^2 x}{\cos^2 x + \cos x \sin x}$.

11. $\dfrac{1 - \tan^2 x}{\tan x + 1}$.

12. $\dfrac{\sin^2 x - 1}{(\sin x - 1)^2}$.

Group B

13. $\dfrac{a^3 - a^2b}{ab - a^2}$.

14. $\dfrac{(a^2 + b^2)^2}{a^4 - b^4}$.

15. $\dfrac{.09x^2 - .16y^2}{.6x - .8y}$.

16. $\dfrac{x^4 - y^4}{x^2 - y^2}$. **17.** $\dfrac{x^{2n} - y^{2n}}{x^n + y^n}$. **18.** $\dfrac{x^2 - y^4}{y^4 - 2xy^2 + x^2}$.

19. $\dfrac{\frac{1}{2}ab\cos C - \frac{1}{2}ab_1\cos C}{\frac{1}{2}ab\cos C - \frac{1}{2}a_1b\cos C}$.

Group C

20. $\dfrac{(a+y)^2 - 4z^2}{a^2 - (y+2z)^2}$. **21.** $\dfrac{y^3 + 6y^2 + 11y + 6}{y^3 - 7y - 6}$.

22. $\dfrac{(x-y)(x+y)(x-z)}{(y+x)(y-x)(z-x)}$.

Recall fact 29. *The product of two or more fractions is a fraction whose numerator is the product of the numerators of the given fractions and whose denominator is the product of the denominators of the given fractions.*

Recall fact 30. *To multiply fractions:*

1. *Factor all the numerators and denominators.*

2. *Remove all factors common to any numerator and any denominator by dividing that numerator and denominator by such common factors.*

3. *Multiply the remaining factors of the numerators to make the new numerator and multiply the remaining factors of the denominators to make the new denominator.*

Recall fact 31. *To divide one fraction by another, invert the divisor and multiply the two fractions.*

Illustrative example. Simplify

$$\frac{x^2 - 16}{x^2 - 9} \cdot \frac{x^2 - 6x + 9}{x^2 - 5x + 4} \div \frac{x - 3}{x^2 - x}.$$

Solution

Invert the divisor and factor.

$$\frac{(x+4)\cancel{(x-4)}}{(x+3)\cancel{(x-3)}} \cdot \frac{\cancel{(x-3)}\cancel{(x-3)}}{\cancel{(x-4)}\cancel{(x-1)}} \cdot \frac{x\cancel{(x-1)}}{\cancel{(x-3)}} = \frac{x(x+4)}{(x+3)}.$$

Check

$$\frac{x^2-16}{x^2-9}\cdot\frac{x^2-6x+9}{x^2-5x+4}\div\frac{x-3}{x^2-x}=\frac{x(x+4)}{(x+3)}.$$

Let $x = 5.$ $$\frac{25-16}{25-9}\cdot\frac{25-30+9}{25-25+4}\div\frac{5-3}{25-5}=\frac{5(5+4)}{(5+3)}.$$

$$\tfrac{9}{16}\cdot\tfrac{4}{4}\cdot\tfrac{20}{2}=\tfrac{45}{8}=\tfrac{45}{8}.$$

EXERCISES

Group A

Simplify:

1. $\dfrac{a^2m}{b^2c}\cdot\dfrac{x^3}{a^3}\cdot\dfrac{ac^2}{mx^2}.$

3. $\dfrac{x^2-4}{x^2-3x+2}\div\dfrac{x^2-2x}{x^3-x}.$

2. $\dfrac{a^2+8a+16}{a^2-9}\cdot\dfrac{a-3}{a+4}.$

4. $\dfrac{R_1^2+R_2^2}{R_1}\div\dfrac{R_1^4-R_2^4}{R_1^2}.$

5. $\dfrac{24x^2}{x^3+3x^2}\cdot\dfrac{x^2+6x+9}{3x}\div\dfrac{8(x+3)}{x}.$

6. $\dfrac{2-3x}{3x}\cdot\dfrac{9x^2y}{4-9x^2}\div\dfrac{4y^3}{2+3x}.$

7. $\dfrac{2\sin A}{5\cos A}\cdot\dfrac{10\cos^3 A}{4\sin^3 A}.$

8. $\dfrac{\sin^2 x}{\sin^2 x-\sin x}\cdot\dfrac{\sin x-1}{2\sin x}.$

9. $\dfrac{\sin^2 x-\cos^2 x}{\cos x}\div\dfrac{(\sin x-\cos x)^2}{\cos x}.$

10. $\dfrac{\tan^2 A}{\tan^2 A-1}\cdot\dfrac{\tan A-1}{1}\div\dfrac{\tan^2 A}{\tan A+1}.$

Group B

11. $\dfrac{x^2-.09}{x^2-3x+9}\cdot\dfrac{.2x}{.06+.2x}\div\dfrac{.6x-2x^2}{9-3x+x^2}.$

12. $(3x^2-7xy+2y^2)\left(\dfrac{2x}{x^2+3xy-10y^2}\right)\div\left(\dfrac{3x-y}{y}\right)^2.$

13. $\dfrac{a^2 - a}{a^2 + 2a + 1} \div \dfrac{1 - 2a + a^2}{1 + a}.$

14. $\dfrac{\sin^2 x - \sin^3 x}{\sin^2 x + \sin^3 x} \div (\sin^2 x - 1).$

Group C

15. $(x^3 + 3x^2 - 6x - 8) \div \dfrac{x^2 + 2x - 8}{x + 1}.$

16. $\dfrac{x^3 - 6x^2 + 36x}{x^4 + 216x} \div \dfrac{x^2 - 49}{x^2 - x - 42}.$

17. $\dfrac{a^m b^n}{4x} \cdot \dfrac{6x^2}{a^{m+1} b^{2n}} \div \dfrac{3}{b^{3n}}.$

18. $\dfrac{x^{n+1}}{3x} \cdot \dfrac{5x^3}{6x^n} \div \dfrac{4x}{3x^6}.$

Recall fact 32. *If fractions are added or subtracted, their sum or their difference must be expressed as* one *fraction. As in arithmetic, fractions may be added or subtracted only when they have like denominators.*

To add or subtract fractions:

1. *Arrange the denominators in order. Use recall fact 27 if necessary.*

2. *Factor the denominators and obtain the lowest common denominator.*

3. *Change the fractions into equivalent fractions having the lowest common denominator as their denominators.*

4. *Add or subtract the numerators as indicated and place the result over the lowest common denominator.*

5. *Simplify the numerator and combine similar terms.*

6. *Factor the numerator if possible and reduce the fraction to lowest terms.*

Illustrative examples.

Example 1. Combine $\frac{4y+1}{3} - \frac{6y-1}{9} - \frac{4y+2}{6}$.

Analysis

Which recall fact must be applied to change these fractions into equivalent fractions all having equal denominators? What is the smallest number of which each of these three denominators is a factor? To get an equivalent fraction having 18 for its denominator by what number must we multiply the terms of the first fraction? of the second fraction? of the third?

Solution

$$\frac{4y+1}{3} - \frac{6y-1}{9} - \frac{4y+2}{6}$$
$$= \frac{6(4y+1)}{18} - \frac{2(6y-1)}{18} - \frac{3(4y+2)}{18}$$
$$= \frac{24y+6-12y+2-12y-6}{18}$$
$$= \tfrac{2}{18} = \tfrac{1}{9}.$$

Check

Let $y = 2$.

$$\frac{4y+1}{3} - \frac{6y-1}{9} - \frac{4y+2}{6}$$
$$= \tfrac{9}{3} - \tfrac{11}{9} - \tfrac{10}{6}$$
$$= \tfrac{1}{9}.$$

Example 2. Combine and simplify

$$\frac{5y}{y-9} - \frac{6y}{y-3} + \frac{y^2-30y-27}{y^2-12y+27}.$$

Solution

$$\frac{5y}{y-9} - \frac{6y}{y-3} + \frac{y^2-30y-27}{(y-9)(y-3)}.$$

$$\text{L. C. D.} = (y-9)(y-3).$$

$$= \frac{5y(y-3) - 6y(y-9) + (y^2-30y-27)}{(y-9)(y-3)}$$
$$= \frac{5y^2-15y-6y^2+54y+y^2-30y-27}{(y-9)(y-3)}$$
$$= \frac{9y-27}{(y-9)(y-3)} = \frac{9(y-3)}{(y-9)(y-3)} = \frac{9}{(y-9)}.$$

EXERCISES

Group A

Combine and simplify:

1. $\frac{a}{5} - \frac{2b}{3} + \frac{2a}{15}$.

2. $\frac{6a + 5b}{10} - \frac{2a - 5b}{6}$.

3. $\frac{b - c}{bc} - \frac{a - c}{ac}$.

4. $\frac{5}{x} + \frac{2}{x - y} - \frac{3y}{x(x - y)}$.

5. $\frac{x - 2}{x + 2} + \frac{x + 2}{x - 2}$.

6. $\frac{5}{3x - 3} - \frac{8}{5x - 15}$.

7. $\frac{a + b}{a - b} - \frac{a - b}{a + b}$.

8. $\frac{y + 33}{y^2 - 9} - \frac{6}{y - 3} + \frac{10}{y + 3}$.

9. $\frac{b}{a + b} - \frac{a}{b - a} - \frac{a^2 + b^2}{a^2 - b^2}$.

10. $\frac{\cos a}{\sin a} + \frac{\sin a}{\cos a}$.

11. $\frac{\cot B}{2} - \frac{1}{2\cot B}$.

12. $\tan A + \frac{1}{\tan A}$.

13. $\frac{1 + \cos A}{2} - \frac{\sin A}{1 - \cos A}$.

14. $\frac{a}{\sin A} + \frac{b}{\sin B} + \frac{c}{\sin C}$.

Group B

15. $\frac{2}{y-1} - \frac{3}{y-2} - \frac{4}{y^2 - 3y + 2}$.

16. $\frac{5 - 2x}{(x-1)^3} + \frac{x+1}{(x-1)^2} + \frac{1}{1-x}$.

17. $2a - 1 + \frac{-8a^3 + 1}{4a^2 - 4a + 1}$.

18. $\frac{1 - \cos A}{1 + \cos A} - \frac{\cos^2 A}{1 - \cos^2 A}$.

19. $\frac{2\cos x}{\cos^2 x - \sin^2 y} + \frac{1}{\cos x - \sin y} - \frac{1}{\cos x + \sin y}$.

Group C

20. $\frac{4}{(x - 1)(x - 3)} + \frac{2}{(2 - x)(x - 1)} + \frac{2}{(x - 2)(3 - x)}$.

21. $\frac{y + x}{y - x} - 2\left(\frac{y}{x} - \frac{y}{x - y}\right)$.

22. $\frac{\frac{1}{9}x^2 - \frac{1}{4}y^2}{2x^2 + 3xy} - \frac{1}{18}$.

Complex fractions. A complex fraction is a fraction having one or more fractions in either or both its numerator and denominator. Just as the simple fraction $\frac{a}{b}$ means $a \div b$, so the complex fraction

$$\frac{\frac{x}{y} + \frac{y}{x}}{x - \frac{1}{x}} \text{ means } \left(\frac{x}{y} + \frac{y}{x}\right) \div \left(x - \frac{1}{x}\right).$$

Observe that every complex fraction can be put in this form.

You will remember from your study of elementary algebra that $\left(x - \frac{1}{x}\right)$ is a mixed expression since it contains the difference (or sum) of a fractional term and a non-fractional term.

Illustrative example. Simplify $\dfrac{\frac{1}{x} - \frac{1}{y}}{\frac{y^2}{x} - x}$.

Solution

Write the complex fraction in the division form suggested above.	$\left(\frac{1}{x} - \frac{1}{y}\right) \div \left(\frac{y^2}{x} - x\right)$
Perform the work indicated within each parenthesis separately.	$= \frac{y - x}{xy} \div \frac{y^2 - x^2}{x}$
Invert the divisor and factor.	$= \frac{y - x}{xy} \cdot \frac{x}{(y + x)(y - x)}$
Reduce to lowest terms.	$= \frac{1}{y(y + x)}$.

$$\therefore \frac{\frac{1}{x} - \frac{1}{y}}{\frac{y^2}{x} - x} = \frac{1}{y(y + x)}.$$

Check

Let $x = 2$, $y = 1$. $\left(\frac{1}{2} - \frac{1}{1}\right) \div \left(\frac{1^2}{2} - 2\right) = \frac{1}{1(1 + 2)}$.

$$(-\tfrac{1}{2}) \div (-\tfrac{3}{2}) = \tfrac{1}{3}.$$

$$\tfrac{1}{3} = \tfrac{1}{3}.$$

EXERCISES

Group A

Simplify each of the following:

1. $\dfrac{a + \dfrac{1}{b}}{a - \dfrac{1}{b}}$.

2. $\dfrac{\dfrac{14}{x^3}}{\dfrac{7}{x} - 14}$.

3. $\dfrac{2x + \frac{1}{3}}{4x^2 - \frac{1}{9}}$.

4. $\dfrac{4xy - 1}{4xy - \dfrac{1}{4xy}}$.

5. $\dfrac{\dfrac{2y^2 + 2x^2}{y - x}}{\dfrac{y^2 + x^2}{y^2 - x^2}}$.

6. $\dfrac{\dfrac{2}{x} + 2 + \dfrac{x}{2}}{\dfrac{4}{x} - x}$.

7. $\dfrac{\dfrac{2x^2 - 4x}{x + 2}}{\dfrac{x^2 - 4}{2x + 4}}$.

8. $\dfrac{\dfrac{y}{x - y} + 1}{\dfrac{y}{x + y} - 1}$.

9. $\dfrac{\dfrac{x^2}{a} + a}{\dfrac{a^3}{x} - \dfrac{x^3}{a}}$.

10. $\dfrac{1 - \dfrac{1}{y + 1}}{1 + \dfrac{1}{y - 1}}$.

11. $\left(\dfrac{15}{a^2} - \dfrac{8}{a} + 1\right) \div \left(1 - \dfrac{5}{a}\right)$.

12. $\left(4 - \dfrac{5}{x+1}\right) \div \left(16 + \dfrac{15}{x^2 - 1}\right)$.

13. $\dfrac{\dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A}}{\dfrac{1}{\sin A \cos A}}$.

 Check, using $A = 45°$.

14. $\dfrac{\tan A + \dfrac{1}{\tan A}}{\tan A - \dfrac{1}{\tan A}}$.

15. $\dfrac{\dfrac{1}{\cos^2 A} - \dfrac{1}{\sin^2 A}}{\dfrac{1}{\cos A} - \dfrac{1}{\sin A}}$.

16. $\dfrac{\cot B - \dfrac{1}{\tan A}}{\tan B + \dfrac{1}{\cot B}}$.

17. $\left(\sin A - \dfrac{\sin A}{\tan B}\right) \div \left(\dfrac{\cos A}{\tan B} - \cos A\right)$.

Group B

18. $$\frac{\dfrac{x^2}{x+2y}-y}{\dfrac{x-4}{2x}-\dfrac{y-2}{2y}}.$$

19. $$\frac{\dfrac{x+y}{x^2+y^2}-\dfrac{1}{x+y}}{\dfrac{1}{x+y}-\dfrac{x}{x^2+y^2}}.$$

20. $$\left(\frac{2y}{y-2}-\frac{y}{y-1}\right)\div\left(\frac{3y}{y-3}-\frac{2y}{y-2}\right).$$

Group C

21. $$\left(\frac{5}{y^2-4}+\frac{2}{2-y}\right)\div\left(2-\frac{3}{2-y}\right).$$

22. $$\left(\frac{x+7}{x-3}+\frac{12}{x^2-3x}\right)\left(\frac{x^2+3x}{2x+8}+\frac{1}{x+4}\right)\div\left(x+5+\frac{6}{x}\right).$$

23. $$\left(1+\frac{x^2}{1-x^2}\right)\left(2x+\frac{2}{x+1}\right)\div\left(\frac{1}{1+x}+\frac{1}{1-x}\right).$$

24. $$\frac{76.56\times\dfrac{8491}{100}-\dfrac{7656}{100}\times 35.72}{49.19\times\dfrac{634}{1000}+\dfrac{566}{1000}\times 49.19}.$$

25. $$\frac{\left(\dfrac{1}{\cos B}+\dfrac{1}{\sin A}\right)}{\left(\dfrac{1}{\tan C}-1\right)\div\left(\sin A+\dfrac{\cos B+\sin A\tan^2 C}{1-\tan^2 C}\right)}.$$

A Study in Changes

1. Complete each statement in this exercise by using only one of the following: increased, decreased, unchanged, impossible to tell.

(*a*) In $\frac{x}{y}$, if x only is increased, then $\frac{x}{y}$ is ? .

(*b*) In $\frac{x}{y}$, if y only is increased, then $\frac{x}{y}$ is ? .

(*c*) In $\frac{x}{y}$, if x and y are both increased, then $\frac{x}{y}$ is ? .

(*d*) If x and y are both doubled, then $\frac{x}{y}$ is ? .

(*e*) If x is increased, $\frac{x}{x-1}$ is ? .

(*f*) If x is increased, $\frac{x}{x+1}$ is ? .

(*g*) If x is increased, then $\frac{1}{\sin x}$ is ? .

(*h*) If x is increased, then $\frac{1}{\cos x}$ is ? .

(*i*) If x is increased, then $\frac{\sin x}{\cos x}$ is ? .

(*j*) If x is increased, then $\frac{\cos x}{\tan x}$ is ? .

2. As x increases, what change takes place in the value of:

(*a*) $\frac{1}{x}$? (*b*) $1+\frac{1}{x}$? (*c*) $1-\frac{1}{x}$?

(*d*) $\left(1+\frac{1}{x}\right) \div \left(1-\frac{1}{x}\right)$? (*e*) $\left(1+\frac{1}{\sin x}\right) \div \left(1-\frac{1}{\sin x}\right)$?

3. (*a*) If $y = \frac{4-x}{2}$, does y increase or decrease as x increases from $+1$ to $+4$?

(*b*) If $y = \frac{4-\cos x}{2}$, does y increase or decrease as x increases from 0° to 90°?

4. (*a*) If x is increased, then its reciprocal is ? .

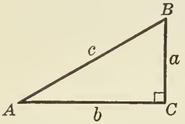
(*b*) If $\tan x$ is increased, then its reciprocal is ? ; in other words, as $\tan x$ increases, $\cot x$? .

5. (*a*) If $y = \frac{7}{1-\frac{1}{x}}$, does y increase or decrease as x increases from $+2$ to $+4$?

(*b*) If x is greater than 1, can y ever be less than 7?

Some New Trigonometric Relations

1. The Pythagorean relations. Using the Pythagorean theorem several new trigonometric relations may be derived from the adjoining diagram.



$a^2 + b^2 = c^2.$	Pythagorean theorem.
$\dfrac{a^2}{c^2} + \dfrac{b^2}{c^2} = \dfrac{c^2}{c^2}.$	Dividing by c^2.
or $\left(\dfrac{a}{c}\right)^2 + \left(\dfrac{b}{c}\right)^2 = 1.$	
But $\dfrac{a}{c} = \sin A$, and $\dfrac{b}{c} = \cos A.$	
$\therefore \sin^2 A + \cos^2 A = 1.$	Substitution.

The sum of the squares of the sine and the cosine of an angle is equal to one.

Also

$$a^2 + b^2 = c^2.$$
$$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}.$$
$$\left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2.$$

But $\dfrac{a}{b} = \tan A$, and $\dfrac{c}{b} = \sec A.$

$\therefore \tan^2 A + 1 = \sec^2 A.$

Again

$$a^2 + b^2 = c^2.$$
$$\frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2}.$$
$$1 + \left(\frac{b}{a}\right)^2 = \left(\frac{c}{a}\right)^2.$$

But $\dfrac{b}{a} = \cot A$, and $\dfrac{c}{a} = \csc A.$

$\therefore 1 + \cot^2 A = \csc^2 A.$

Express in words the relations between the squares of the tangent and secant of an angle; the squares of the cotangent and cosecant.

Pythagorean Relations

$$\sin^2 A + \cos^2 A = 1.$$
$$\tan^2 A + 1 = \sec^2 A.$$
$$\cot^2 A + 1 = \csc^2 A.$$

2. **The quotient relations.** We know from the diagram on page 72 that:

$$\sin A = \frac{a}{c}.$$

$$\cos A = \frac{b}{c}.$$

$$\therefore \frac{\sin A}{\cos A} = \frac{\frac{a}{c}}{\frac{b}{c}}$$

$$= \frac{a}{b}$$

$$= \tan A$$

$$\text{or } \tan A = \frac{\sin A}{\cos A}.$$

$$\cos A = \frac{b}{c}.$$

$$\sin A = \frac{a}{c}.$$

$$\therefore \frac{\cos A}{\sin A} = \frac{\frac{b}{c}}{\frac{a}{c}}$$

$$= \frac{b}{a}$$

$$= \cot A$$

$$\text{or } \cot A = \frac{\cos A}{\sin A}.$$

Express the quotient relations in words.

Quotient Relations

$$\tan A = \frac{\sin A}{\cos A}.$$

$$\cot A = \frac{\cos A}{\sin A}.$$

3. **The reciprocal relations.** You will remember that in Chapter I we developed certain reciprocal relations, namely:

$$\sin A \times \csc A = 1 \quad \text{or} \quad \csc A = \frac{1}{\sin A}.$$

$$\cos A \times \sec A = 1 \quad \text{or} \quad \sec A = \frac{1}{\cos A}.$$

$$\tan A \times \cot A = 1 \quad \text{or} \quad \cot A = \frac{1}{\tan A}.$$

Express the reciprocal relations in words.

Identities

Since the trigonometric relations were established without regard to the number of degrees contained in acute angle A, they are true for all values of the angle between $0°$ and $90°$. A relation which is true for all values of the letter or letters involved (and for which both sides have a meaning) is called an *identity.* Hence the Pythagorean, the quotient, and the reciprocal relations are *trigonometric identities.*

A relation which is true only for certain or restricted values of the letters involved is called an *equation of condition* or just an equation.

Something to think about. Of the following, which are equations? Which are identities? Explain why.

1. $(a+b)^2 = a^2 + 2ab + b^2$.
2. $4x - 5 = 20$.
3. $x^2 - 5x + 6 = 0$.
4. $x + y = 7$.
5. $x^2 - 5x + 6 = (x - 3)(x - 2)$.
6. $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.
7. $A = bh$.
8. $c = \pi r^2$.
9. $\sin A = \frac{1}{2}$.
10. $\sin^2 x = 1 - \cos^2 x$.
11. $I = \text{prt}$.
12. $\sin A = \cos B$.
13. $\tan y = 0$.
14. $\sec A = \sin B$.

How identities are proved to be true. Since identities are true for all values of the letters involved, it follows that both sides or members of the identity must always have the same value. Therefore, it ought to be, and it is, possible to transform either side to the exact form of the other or to transform both sides into forms that are identical. When this is done, the identity is said to be proved.

The following illustrative examples will show the three possible methods of proving an identity.

Illustrative examples.

Example 1. Prove the identity $\cos A\ (\cot A + \tan A) = \csc A$ by transforming the left-hand member into the right-hand member.

Solution

Substituting $\dfrac{\cos A}{\sin A}$ for $\cot A$ and $\dfrac{\sin A}{\cos A}$ for $\tan A$.	$\cos A(\cot A + \tan A)$ $= \cos A \left(\dfrac{\cos A}{\sin A} + \dfrac{\sin A}{\cos A}\right)$
Combining fractions within the parenthesis.	$= \cos A\ \dfrac{(\cos^2 A + \sin^2 A)}{\sin A \cos A}$
Substituting 1 for $\cos^2 A + \sin^2 A$.	$= \cos A\ \dfrac{1}{\sin A \cos A}$
Multiplying and reducing.	$= \dfrac{1}{\sin A}$
Substituting $\csc A$ for $\dfrac{1}{\sin A}$.	$= \csc A.$

$$\therefore\ \cos A(\cot A + \tan A) = \csc A.$$

Example 2. Prove the identity $\sin A = \cot A \cos A - \csc A\ (1 - 2 \sin^2 A)$ by transforming the right-hand member into the left-hand member. Explain each step of the solution.

Solution

$$\begin{aligned}
\cot A \cos A - \csc A\ (1 - 2\sin^2 A) \\
&= \frac{\cos A}{\sin A} \cdot \cos A - \frac{1}{\sin A}\ (1 - 2\sin^2 A) \\
&= \frac{\cos^2 A}{\sin A} - \frac{1}{\sin A} + \frac{2\sin^2 A}{\sin A} \\
&= \frac{\cos^2 A - 1 + 2\sin^2 A}{\sin A} \\
&= \frac{1 - \sin^2 A - 1 + 2\sin^2 A}{\sin A} \\
&= \frac{\sin^2 A}{\sin A} = \sin A.
\end{aligned}$$

$$\therefore\ \sin A = \cot A \cos A - \csc A(1 - 2\sin^2 A).$$

Example 3. Prove the identity $\dfrac{\sec A - \csc A}{\sec A + \csc A} = \dfrac{\tan A - 1}{\tan A + 1}$ by transforming both sides until you obtain two expressions that are identical. Explain each step of the solution.

Solution

$$\frac{\sec A - \csc A}{\sec A + \csc A} = \frac{\tan A - 1}{\tan A + 1} \tag{1}$$

if
$$\frac{\dfrac{1}{\cos A} - \dfrac{1}{\sin A}}{\dfrac{1}{\cos A} + \dfrac{1}{\sin A}} = \frac{\dfrac{\sin A}{\cos A} - 1}{\dfrac{\sin A}{\cos A} + 1} \tag{2}$$

if
$$\left(\frac{1}{\cos A} - \frac{1}{\sin A}\right) \div \left(\frac{1}{\cos A} + \frac{1}{\sin A}\right) = \left(\frac{\sin A}{\cos A} - 1\right) \div \left(\frac{\sin A}{\cos A} + 1\right) \tag{3}$$

if
$$\frac{\sin A - \cos A}{\sin A \cos A} \div \frac{\sin A + \cos A}{\sin A \cos A} = \frac{\sin A - \cos A}{\cos A} \div \frac{\sin A + \cos A}{\cos A} \tag{4}$$

if
$$\frac{\sin A - \cos A}{\sin A + \cos A} = \frac{\sin A - \cos A}{\sin A + \cos A}. \tag{5}$$

Since (5) is true, (4) is true. Why? But (4) is derived from (3), so (3) must be true, and so on back to (1) which is true because all the steps are reversible.

Notice the similarity between the kind of reasoning used in proving this identity and the reasoning used in the analytic method employed in the study of geometry.

Something to think about.

1. Can identities always be reduced to the form $1 = 1$?

2. In proving an identity, why is it better, when possible, to express all functions in terms of sines and cosines?

The trigonometric relations will enable you to express all functions in terms of a single function as either the sine or the cosine. If this cannot be done simply, express all the functions in the identity to be proved in terms of sines and cosines.

Remember that there are many ways of proving an identity, and that the shortest proof is the one preferred.

EXERCISES

Group A

Prove the following identities:

1. $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$.

2. $\dfrac{x^2 - 3xy + 2y^2}{x - y} = x - 2y$.

3. $\dfrac{a}{a-b} - \dfrac{a^2}{a^2 - b^2} = \dfrac{b}{a+b} + \dfrac{b^2}{a^2 - b^2}$.

4. $\tan A \cos A = \sin A$.

5. $\sin B \cot B = \cos B$.

6. $\sec B \cot B = \csc B$.

7. $\csc A \tan A = \sec A$.

8. $\cos A \csc A = \cot A$.

9. $\sin B \sec B = \tan B$.

10. $\sin A \sec A \cot A = 1$.

11. $\cos B \tan B \csc B = 1$.

12. $\sec x - \tan x \sin x = \cos x$.

13. $\sec A \csc A = \cot A + \tan A$.

14. $\tan A \sin A + \cos A = \sec A$.

15. $\sec A - \cos A = \tan A \sin A$.

16. $\cos^2 A(1 + \tan^2 A) = 1$.

17. $\sec^2 y = 1 + \sec^2 y \sin^2 y$.

18. $\cos^2 A - \sin^2 A = 2\cos^2 A - 1$.

19. $(1 - \sin^2 A)\csc^2 A = \cot^2 A$.

20. $(\tan\theta + \cot\theta)^2 = \sec^2\theta + \csc^2\theta$.
    (θ, a Greek letter, called theta)

21. $\dfrac{1}{\sec^2 A} + \dfrac{1}{\csc^2 A} = 1$.

22. $\dfrac{1 - \cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1 + \cos\theta}$.

23. $\dfrac{\sin\theta}{1 - \cos^2\theta} = \csc\theta$.

24. $\dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A} = \sec A \csc A$.

Group B

25. $\cot^2 A - \cos^2 A = \cot^2 A \cos^2 A.$

26. $\cos^4 x - \sin^4 x = \cos^2 x - \sin^2 x.$

27. $\tan x + \tan y = \tan x \tan y\,(\cot x + \cot y).$

28. $\sin^2 A \cos^2 A + \cos^4 A = 1 - \sin^2 A.$

29. $(\sin A + \cos A)^2 + (\sin A - \cos A)^2 = 2.$

30. $\cot A + \dfrac{\sin A}{1 + \cos A} = \csc A.$ **31.** $\dfrac{\tan A - 1}{\tan A + 1} = \dfrac{1 - \cot A}{1 + \cot A}.$

32. $\dfrac{\csc B}{\cot B + \tan B} = \cos B.$

33. $2 \sin A \cos A = \dfrac{2 \tan A}{1 + \tan^2 A}.$

34. $(\tan \theta + \sec \theta)^2 = \dfrac{1 + \sin \theta}{1 - \sin \theta}.$

Group C

35. $\sec^4 A - \tan^4 A = 1 + 2 \tan^2 A.$

36. $(\sin^2 x - \cos^2 x)^2 = 1 - 4 \sin^2 x \cos^2 x.$

37. $(1 - \tan^2 A)^2 + 4 \tan^2 A = \sec^4 A.$

38. $\sin^3 A \cos A + \cos^3 A \sin A = \sin A \cos A.$

39. $\tan^2 A + \tan^4 A = \dfrac{\sin^2 A}{\cos^4 A}.$

40. $\dfrac{\tan A}{1 - \tan^2 A} = \dfrac{\sin A \cos A}{\cos^2 A - \sin^2 A}.$

41. $\dfrac{\cos A}{1 - \tan A} + \dfrac{\sin A}{1 - \cot A} = \sin A + \cos A.$

42. $\dfrac{(1 - \sin A - \cos A)^2}{2} = (1 - \cos A)(1 - \sin A).$

43. $\dfrac{\sin^2 x}{\cot^2 x} + \dfrac{\cos^2 x}{\tan^2 x} + 1 = \tan^2 x + \cot^2 x.$

44. $\left(\dfrac{\sec A + \csc A}{1 + \tan A}\right)^2 = \dfrac{\tan A + \cot A}{\tan A}.$

Fractions in Early Times

The "Ahmes Papyrus" contains some very interesting information on the methods employed by the Egyptians in fractions. The Egyptians made extensive use of fractions that had unit numerators; the Romans confined most of their fraction work to those having 12 for denominators; while the Greeks were concerned more with the Babylonian system of sexagesimal fractions, that is, fractions whose denominators are 60.

The system of writing the numerator above the denominator is attributed to Brahmagupta in the seventh century, and this was later improved upon by the Arabs, who inserted a bar between the two terms of the fraction.

In general, however, all improvements in devices dealing with fractions have been stimulated by the commercial necessity of making the work mechanical.

CUMULATIVE REVIEW

Chapters II, III, and IV

1. Which of these statements are true? Which are false?

(*a*) The value of $a^2 - b$ is the same as the value of $-(a^2 - b)$ when a is greater than b.

(*b*) $4\,a^3x^y$ and $5\,a^yx^3$ are similar terms.

(*c*) The factors of $\sin^2 x - \cos^2 x$ are equal to the factors of $\cos^2 x - \sin^2 x$.

(*d*) $x^2 - x$ is a factor of $x^3 - x$.

(*e*) $m^2 + 4$ cannot be factored and is therefore prime.

(*f*) $\dfrac{2\,b + a}{2} = b + a.$

(*g*) $\dfrac{a - b}{x - y} - \dfrac{b}{y - x} = \dfrac{a}{x - y}.$

(*h*) The relation $\tan x = \dfrac{\sin x}{\cos x}$ is an identity.

(*i*) A fraction is unchanged in value if the same number is added to both its numerator and denominator.

(*j*) In the statement $y = \dfrac{10}{5 + \dfrac{1}{x}}$, for positive values of x, the value of y can never exceed 2.

2. Complete each of the following statements:

(*a*) $a^2 - 2\,ab + b^3 = a^2 - (\ ?\ )$.

(*b*) The expansion of $(2\,x - y)^5$ has ? terms of which ? are negative.

(*c*) The expression of which $a^x - y$ and $a^x + y$ are factors is ? .

(*d*) Two factors of $a^{n+1} - a$ are ? .

(*e*) The binomial factor of $32 + y^5$ is ? .

(*f*) Multiplying a fraction by $\frac{1}{3}$ is equivalent to dividing the fraction by ? .

(*g*) To reduce $\dfrac{\tan^2 A - 1}{2\tan A - 2}$ to lowest terms, you must divide numerator and denominator by ? .

(*h*) The fraction $\dfrac{2\,x + 3}{2\,x - y}$ exceeds the fraction $\dfrac{x + 3}{2\,x - y}$ by ? .

(*i*) $\sec^2 y$ exceeds $\tan^2 y$ by ? .

(*j*) If $\tan A \times \cot B = 1$, $A =$? degrees.

3. Subtract $ax^2 - by^2$ from $px^2 - qxy + ry^2$ and write the result as a trinomial having an x^2, xy, and y^2 term.

4. The area of a rectangle is $6\,y^3 + 11\,y^2 - 1$ and its base is $2\,y^2 + 3\,y - 1$. Find its altitude and perimeter.

5. Show that the three factors of the expression $(2\,a + b)^2 - (a + 2\,b)^2$ are the same as the three factors of $3\,a^2 - 3\,b^2$.

6. (*a*) Factor: $y^3 + 7\,y^2 - 36$.

(*b*) For negative values of y, which factor will always have the least value?

7. Solve by factoring and check: $3\,x^2 = 22\,x - 7$.

8. Combine and check: $1 + \dfrac{2x}{2x - 1} + \dfrac{8x^2 - 2}{1 - 4x^2}$.

9. Simplify: $\dfrac{x - \dfrac{4}{2 - x} - 7}{x - \dfrac{7}{x - 2} + 4}$.

10. Prove the identity $\sin x \tan x + \cos x = \sec x$.

11. Reduce to lowest terms: $\dfrac{4 \tan^2 x - 2 \tan x}{4 \tan^2 x - 4 \tan x + 1}$.

12. (*a*) The perimeter of a triangle is $\dfrac{150}{y^2 - 3y + 2}$ and two of its sides are $\dfrac{6y}{y - 2}$ and $\dfrac{6y}{y - 1}$. Find the third side.

(*b*) Find the three sides of the triangle when $y = 4$.

CHAPTER V. SOLVING EQUATIONS

Equations always balance, and he, who has once comprehended this fact as a result of much manipulation of equations, is informed.

— CHARLES H. JUDD.

IDENTICAL AND CONDITIONAL EQUATIONS

Try this puzzle on your friends: A bottle and a cork cost \$1.10. The bottle costs \$1 more than the cork. What is the cost of the bottle?

Of course, a student could probably arrive at the answer by trial and error, but algebra provides a definite way by which he may obtain the result with ease and precision.

Recall fact 33. *An equation is a statement of the equality of two algebraic expressions. The expressions are called the sides or members of the equation.*

Equations are of two kinds — identical equations and conditional equations.

Recall fact 34. *An identical equation, or identity, is true for all values of the letters concerned (for which both sides of the equation have a meaning).*

Recall fact 35. *A conditional equation is true only for particular or certain values of the letters concerned.*

Notice that in $(a + b)(a - b) = a^2 - b^2$, the equation is true for all values of a and b, but that in $2x - 5 = 13$ the equation is true only when $x = 9$. Which of these is an identical equation (identity)? which a conditional equation?

Recall fact 36. *When we check the solution of a conditional equation by substitution, we get an identity. That quantity which, when substituted for the unknown letter in this conditional equation, reduces it to an identity, is said to satisfy the equation.*

Recall fact 37. *That quantity which satisfies a conditional equation is called a root of the equation.*

Recall fact 38. *An equation in which the highest exponent of the unknown quantity or letters is 1 and in which no term contains more than one unknown letter, is called a linear equation.*

Recall fact 39. *Equations different in form but having the same roots are called equivalent equations.*

Recall fact 40. *The solution of an equation is the process of finding the root or roots.*

Certain axioms are used to solve equations; these are:

1. *If the same number is added to each member of an equation, the resulting members are still equal.*
2. *If the same number is subtracted from each member of an equation, the resulting members are still equal.*
3. *If each member of an equation is multiplied by the same number, the resulting members are still equal.*
4. *If each member of an equation is divided by the same number, the resulting members are still equal.*

Recall fact 41. *A term may be removed from one member of an equation to the other, provided its sign is changed. (This transposing of a term is fundamentally its addition to or subtraction from each member of the equation. We transpose terms so as to get all of the unknown quantities in one member and all known quantities in the other.)*

Recall fact 42. *The general procedure in solving a linear equation is:*

1. *Remove all parentheses.*
2. *Transpose all terms containing the unknown quantity to one member and those terms not containing the unknown to the other member.*
3. *Combine similar terms.*
4. *Find the value of the unknown quantity.*
5. *Check in the original equation by substitution.*

Illustrative examples.

Example 1. Solve

$$(3x + 2)(x - 1) - 3(x + 3)(x - 2) = 2x - 26.$$

Solution

$$\begin{aligned}(3x + 2)(x - 1) - 3(x + 3)(x - 2) &= 2x - 26.\\ (3x^2 - x - 2) - 3(x^2 + x - 6) &= 2x - 26.\\ 3x^2 - x - 2 - 3x^2 - 3x + 18 &= 2x - 26.\\ 3x^2 - 3x^2 - x - 3x - 2x &= 2 - 18 - 26.\\ -6x &= -42.\\ x &= 7.\end{aligned}$$

Check

Substituting $x = 7$.

$$\begin{aligned}(21 + 2)(7 - 1) - 3(7 + 3)(7 - 2) &= 14 - 26.\\ 138 - 150 &= -12.\\ -12 &= -12.\end{aligned}$$

Example 2. Solve $3 - (4 \tan x - 5) = 6 \tan x - 2.$

Solution

$$\begin{aligned}3 - 4\tan x + 5 &= 6\tan x - 2.\\ -4\tan x - 6\tan x &= -3 - 5 - 2.\\ -10\tan x &= -10.\\ \tan x &= 1.\\ x &= 45°.\end{aligned}$$

Check by substituting $x = 45°$.

EXERCISES

Solve the following equations and check:

1. $5x - 2x + 12 = 35 - 4x - 9$.
2. $4(4a - 1) + 3 = 2(3 + a)$.
3. $5(2x + 7) = 3(2x - 3)$.
4. $x(x + 5) - 6(x - 3) = 6 - x(2 - x)$.
5. $(x - 2)(x + 3) = (x + 4)(x - 6)$.
6. $(x - 4)(x - 3) = (x + 4)(x - 6) + 1$.
7. $(x - 3)^2 - (x - 2)^2 = -5$.
8. $\tan x - 2 = 3 - 4\tan x$.
9. $16 + 8\cos A - 14 = 12\cos A$.
10. $3(\sin x + 1) = 4 + \sin x$.
11. $4\cos x = 5 - 2(\cos x + 1)$.
12. $4(4 + \tan x) + 5(\tan x - 5) = 0$.
13. $(\tan x - 3)(\tan x - 6) = (\tan x + 4)(\tan x + 1)$.
14. $(\cos x - 3)(\cos x + 1) = (\cos x - 1)(\cos x + 3)$.
15. $(\sin x - 7)(5 - \sin x) + (\sin x - 5)(\sin x + 7) + 70 = 0$.

Solve correct to the nearest tenth:

16. $7(x + 3) - 2(2x + 5) = 7$.
17. $6x(x + 4) - 6(x^2 - 3) = 6 - x$.
18. $5(m + 4) = 13$.
19. $4(4x - 7) = 3(2x + 5)$.
20. Is 4 the root of the equation $5(x + 5) - 53 = 4(x - 6)$?
21. Is .2 the root of the equation $20x + 2 = 4$?
22. Does $(x + 3)(x - 3) = x^2 - 9$, when $x = 0$? when $x = 1$? when $x = 2$? when $x = 9$? Is it therefore an identity or a conditional equation?
23. Does $(x - 2)(x - 2) = x^2$ when $x = 1$? 2? 3? Is it therefore an identity or a conditional equation?

24. Does $\tan x = \dfrac{\sin x}{\cos x}$ when $x = 0°$? $45°$? $60°$? Is it therefore a conditional equation or an identity?

25. Does $\tan x = 1$ when $x = 0°$? $30°$? $45°$? $60°$? $90°$? Is it therefore a conditional equation or an identity?

26. Does $x = 45°$ satisfy the equation

$$5 \tan x - 2 = 2 \tan x + 1?$$

27. Does $x = 60°$ satisfy the equation $2 \cos x - 1 = \sec x - 2$?

Recall fact 43. *To solve an equation containing one or more fractions, multiply each member of the equation by the smallest quantity that will "clear" all terms of fractions. The smallest quantity which will clear the equation of fractions is the lowest common multiple of the denominators.*

Illustrative examples.

Example 1. Solve $\dfrac{2}{x^2 - 4} + \dfrac{5}{x + 2} = \dfrac{2}{x - 2}$.

Solution

$$\frac{2}{(x+2)(x-2)} + \frac{5}{x+2} = \frac{2}{x-2}.$$

The lowest common multiple of the denominators is

$$(x + 2)(x - 2).$$

$$\cancel{(x+2)(x-2)}\frac{2}{\cancel{(x+2)(x-2)}} + \cancel{(x+2)}(x-2)\frac{5}{\cancel{x+2}}$$

$$= (x+2)\cancel{(x-2)}\frac{2}{\cancel{x-2}}.$$

$$2 + 5(x - 2) = 2(x + 2).$$

$$2 + 5x - 10 = 2x + 4.$$

$$3x = 12.$$

$$x = 4.$$

Check by substituting $x = 4$ in the original equation.

Example 2. Solve $\dfrac{6\sin x - 11}{\sin x - 11} - \dfrac{3\sin x - 2}{\sin x + 1} = 3.$

Solution

The lowest common multiple of the denominators is

$$(\sin x - 11)(\sin x + 1).$$

$$\cancel{(\sin x - 11)}(\sin x + 1)\frac{6\sin x - 11}{\cancel{\sin x - 11}}$$

$$-(\sin x - 11)\cancel{(\sin x + 1)}\frac{(3\sin x - 2)}{\cancel{\sin x + 1}} = (\sin x - 11)(\sin x + 1)3.$$

$$(\sin x + 1)(6\sin x - 11) - (\sin x - 11)(3\sin x - 2)$$
$$= 3(\sin x - 11)(\sin x + 1).$$

$$(6\sin^2 x - 5\sin x - 11) - (3\sin^2 x - 35\sin x + 22)$$
$$= 3\sin^2 x - 30\sin x - 33.$$

$$6\sin^2 x - 5\sin x - 11 - 3\sin^2 x + 35\sin x - 22$$
$$= 3\sin^2 x - 30\sin x - 33.$$

$$60\sin x = 0.$$
$$\sin x = 0.$$
$$x = 0°.$$

Check

$$\frac{6(0) - 11}{0 - 11} - \frac{3(0) - 2}{0 + 1} = 3.$$
$$1 + 2 = 3.$$
$$3 = 3.$$

EXERCISES

Solve and check:

1. $\dfrac{x}{3} - \dfrac{x-3}{3} + x = 12 - \dfrac{x+4}{2}.$

2. $\dfrac{1}{4}(3x + 5) + \dfrac{1}{6}(x + 4) = 1.$

3. $\dfrac{1}{3}(S + 3) - \dfrac{5}{6}(S + 1) = \dfrac{2}{3}.$

4. $\dfrac{2x - 5}{x + 6} = \dfrac{1}{9}.$

5. $\dfrac{3}{x + 21} = \dfrac{2}{x - 4}.$

6. $\dfrac{1}{x - 3} + \dfrac{4}{x + 3} = \dfrac{26}{x^2 - 9}.$

7. $\dfrac{10}{x + 4} + \dfrac{1}{4 - x} = \dfrac{10}{x^2 - 16}.$

8. $\dfrac{4\cos x}{5} = \dfrac{2\cos x + 1}{3} - \dfrac{4}{15}.$

9. $\frac{4 \sin x+1}{3}-\frac{2 \sin x+1}{5}=\frac{3}{5}$.

10. $\frac{4}{\tan x}=\frac{5}{\tan x}-1$.

11. $\frac{15}{3-2 \tan x}-\frac{2}{6-4 \tan x}=14$.

12. $\frac{m+3}{2-m}-\frac{3-m}{m+2}=\frac{12}{4-m^2}$.

13. $\frac{x+2}{x+1}-\frac{x-3}{x-1}=\frac{23}{x^2-1}$.

14. $\frac{x-3}{x-1}-\frac{x-1}{x}=\frac{5}{x^2-x}$.

Solve the following correct to the nearest tenth:

15. $\frac{2x}{3}+\frac{x}{4}=\frac{1}{2}$.

16. $\frac{2}{x+3}=\frac{7}{2x-5}$.

17. $\frac{1}{x+3}+\frac{4}{x-3}=\frac{3}{9-x^2}$.

A Study in Changes

The world in which we live is undoubtedly a world of changes, or variations. Frequently there is a close relation between certain changes. There are comparatively few things in life that can be changed without affecting a change in something closely related to them. The examples below give an idea of changes produced by related changes.

Complete each of the following:

1. The number of representatives in Congress allowed each state depends on the ? of the state.

2. The cost of 6 dozen eggs depends upon the ? per dozen.

3. The number of pieces of tile required to tile a bathroom floor depends upon the ? of the room.

4. The cost of sending a package parcel post from New York to Chicago depends upon the ? of the package.

5. The cost of gas for running an automobile depends upon the ? of gallons used and the ? per gallon.

6. The distance a man travels at the rate of 3 miles per hour depends upon the ? .

Changing quantities are called *variables.* In science and mathematics we can determine the result of change when one of two or more closely related quantities are caused to change. In order, therefore, to understand the fluctuations in the world about us and their results, it is very important that we make a study of these relations and changes.

Complete the following:

7. The number of tickets that can be bought for $5 depends upon the ? .

8. The time it takes a boy to ride to school on a bicycle depends upon the ? at which he rides.

9. The length of the circumference of a circle depends upon the length of the ? of the circle.

10. The value of the quantity $2x + 5$ depends upon ? .

11. If the rate at which a man travels is increased, then the time taken to reach his destination is ? (increased or decreased).

12. A poor crop of wheat will result in a ? in the supply and therefore an ? in the price per bushel.

13. The total income derived from an investment depends upon the ? invested, the ? of investment, and the ? . An increase in the investment, the other factors remaining constant, will ? the income.

14. A number of boys agree to buy a boat, each to stand an equal share of the cost. An increase in the number of boys will ? the share of each.

15. The cost of a dress depends upon the ? of material needed; the ? per yard; and the ? of making. An increase in the price per yard will ? the cost of the dress; a decrease in the amount of material needed will ? the cost of the dress.

In each of the following equations x is a positive number. State whether y increases or decreases as (a) x increases and (b) x decreases.

16. $y = x.$

17. $y = x + 1.$

18. $y = 2x.$

19. $x + y = 2.$

20. $y = \frac{1}{x}.$

21. $y = \frac{1}{x} + 1.$

22. $y = 3 - \frac{1}{x}.$

23. $y = \frac{1}{3x - 1}.$

24. $y = \frac{1}{1 - x}.$

25. $y = \frac{1}{1 + \frac{1}{x}}.$

26. $y = x^2.$

27. $y = \frac{3}{x^2}.$

Literal Equations

Recall fact 44. *A literal equation is an equation which involves letters other than the one for which we are solving;* e.g., $ax + bx = c + 5$. *Except in formulas, we generally use the last letters of the alphabet to represent unknown quantities and the first letters to represent known quantities.*

Illustrative example. Solve $ax - am = bx - bm.$

Solution

$$ax - am = bx - bm.$$

Transposing $\quad ax - bx = am - bm.$

Factoring $\quad x(a - b) = am - bm.$

Dividing $\quad x = \frac{am - bm}{a - b}.$

Reducing to lowest terms $\quad x = \frac{m\cancel{(a-b)}}{\cancel{(a-b)}} = m.$

Check

$$ax - am = bx - bm.$$

$$am - am = bm - bm.$$

$$0 = 0.$$

EXERCISES

Solve for the letter indicated in the [ ], otherwise solve for x:

1. $ax + bx = a + b$.
2. $ay + by = c - dy$ [y].
3. $b + x = a - x$.
4. $t - m = mnt - n$ [t].
5. $a(ax + b) = m(ax - b)$.
6. $(a + x)(a + 2x) = (a - 2x)(a - x)$.
7. $(a + x)(a - b) = (a - x)(a + b)$.
8. $\frac{m}{x} = n$.
9. $\frac{b}{x} - \frac{a}{x} = d$.
10. $\frac{2x}{a} + 1 = b$.
11. $\frac{x}{x - a} = \frac{x + b}{x}$.
12. $a = \frac{b + m}{1 + m}$ [m].
13. $\frac{1}{bx} - \frac{a}{b} = \frac{b}{a} - \frac{1}{ax}$.

Changing formulas for computation. Very frequently in the study of physics, chemistry, and the other sciences, we are called upon to make calculations from formulas which express certain relationships.

Illustrative example. In the formula $C = \frac{5(F - 32)}{9}$ solve for F and use the new formula to find the value of F when $C = 20°$.

Solution

$$C = \frac{5(F - 32)}{9}.$$

$$9C = 5(F - 32).$$

$$9C = 5F - 160.$$

$$9C + 160 = 5F.$$

$$\frac{9C + 160}{5} = F.$$

Substituting 20 for C $\quad F = \frac{9(20) + 160}{5}.$

$$F = 68°.$$

Something to think about.

1. How would you check the numerical value of F in the original formula?

2. What happens to C as F is increased?

3. Is the value of C doubled when the value of F is doubled?

4. For what value of F will C equal zero?

EXERCISES

Group A

1. If p represents the principal, r the rate per cent, t the time in years, and A the amount, then the formula for finding the amount in simple interest is $A = p(1 + rt)$.

(a) Solve for p; for r; for t.

(b) Find p when $A = \$345$, $r = 5\%$, and $t = 3$ years.

(c) Find r when $A = \$2500$, $p = \$2000$, and $t = 4$ years.

(d) If the principal is doubled, how does that affect the amount?

(e) If the time is increased, then the amount ? (increases or decreases).

2. $C = 2\,\pi r$ is the formula for the length of the circumference of a circle in terms of the radius.

(a) Solve for r.

(b) Find the value of the radius of a circle whose circumference is 28 ($\pi = \frac{22}{7}$).

(c) What is the effect upon C if r is doubled? trebled? divided by 2?

(d) If r is increased by 1, is C increased by 1?

3. (a) Solve $l = a + (n - 1)d$ for n; for a; for d.

(b) Find n if $l = 49$, $a = 7$, $d = 7$.

4. (a) Solve $S = \dfrac{a}{1 - r}$ for r; for a.

(b) Find r if $a = 1$, $S = 2$.

5. (*a*) Solve $S = \frac{n}{2}(a + l)$ for n; for l; for a.

(*b*) Find n if $S = 196$, $a = 7$, $l = 49$.

6. (*a*) Solve $\frac{a}{\sin A} = \frac{b}{\sin B}$ for a; for b.

(*b*) Find the value of a when $b = 17.2$, angle $A = 35°$, and angle $B = 75°$.

(*c*) If angle A is doubled while B and b remain unchanged, is the value of a doubled? is it increased or decreased?

7. The formula for computing the area K of a parallelogram in terms of two sides a and b and the included angle C is $K = ab \sin C$.

(*a*) Solve for $\sin C$.

(*b*) Find the value of K if $a = 10$, $b = 12$, and angle C is $43°$.

(*c*) What happens to K as C increases from $0°$ to $90°$?

(*d*) For what value of C will the area (K) be a maximum?

Group B

8. The science of physics tells us that if a body near the earth is dropped, the distance (s) in feet that it falls in t seconds may be found from the formula $s = \frac{1}{2}gt^2$.

(*a*) Solve for g; for t.

(*b*) If $t = 5$ seconds, and $g = 32$, find s.

(*c*) If t is doubled, what happens to s?

(*d*) How far will the body have fallen at the end of the third second? (Use $g = 32$.)

(*e*) How long will it take for a body to fall one mile? (Give the answer correct to the nearest tenth of a second.)

9. The formula for computing the area of a trapezoid is $A = \frac{1}{2}h(b + b')$.

(*a*) Solve for b; for h.

(*b*) What is the effect on A if the altitude (h) is doubled? trebled?

10. A formula used in the study of electrical measurements is $C = \dfrac{E}{R + r}$.

(*a*) Solve for R; for r; for E.

(*b*) Calculate R correct to the nearest tenth if $C = .323$, $E = 2.35$, and $r = .3$.

(*c*) If E is increased, while R and r remain constant, how does this affect C?

(*d*) If E and R remain constant, and r increases, how does this affect C?

11. A formula frequently used in physics in the study of lenses is $\dfrac{1}{f} = \dfrac{1}{q} + \dfrac{1}{p}$.

(*a*) Solve this formula for f; for q; for p.

(*b*) Find the value of f when $q = 5$ and $p = 10$.

(*c*) If q does not change, what effect upon f does a constant increase in the value of p produce?

(*d*) What happens to f when $p = q$?

12. If a triangle ABC is inscribed in a circle of radius R, trigonometry teaches us that $R = \dfrac{a}{2 \sin A}$.

(*a*) Solve for a; for $\sin A$.

(*b*) Find the value of R if $A = 46°$, and $a = 3$.

(*c*) As A increases from 0° to 90°, does R increase or decrease?

(*d*) What is the greatest value that $\sin A$ may have? Therefore what is the maximum (greatest) value that $2 \sin A$ may have? Therefore what is the minimum (smallest) value that R may have?

Group C

13. The relationship between the pressure of a gas and its volume is expressed by the formula, $\dfrac{V_1}{V_2} = \dfrac{P_2}{P_1}$.

(*a*) Solve for V_1; V_2; P_2; P_1.

(*b*) Find the value of V_1 when $V_2 = 5$, $P_2 = 32$, $P_1 = 15$.

(*c*) If V_2 and P_2 are kept constant, and P_1 is doubled, is V_1 doubled or halved?

(*d*) If V_2 and P_2 are kept constant, as P_1 increases, V_1 ? .

14. The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

(*a*) Solve for h in terms of the other letters.

(*b*) Find h correct to the nearest tenth if $\pi = 3.14$, $r = 7$, and $V = 62.8$.

(*c*) What happens to V if h is doubled and r is constant? if r is doubled and h is constant? if both h and r are doubled?

(*d*) Which will cause V to change more rapidly, doubling r or doubling h?

15. A formula in physics showing the relation between the weights in the pans of a balance and the lengths of the arms is expressed by the formula $\frac{W_1}{W_2} = \frac{d_2}{d_1}$.

(*a*) Solve for W_1; for W_2; for d_2; for d_1.

(*b*) Find W_1 correct to the nearest tenth if $W_2 = 47.5$, $d_1 = 13.4$, and $d_2 = 5.6$.

(*c*) If the lengths of the arms (d_2, d_1) remain fixed, and W_1 is doubled, what must be done to W_2 to keep the scale in balance?

(*d*) If one weight and the length of its arm (W_1, d_1) remain fixed, and d_2 is halved, what must be done to W_2 to keep the scale in balance?

16. If a, b, and c are the sides of a triangle, the value of angle A can be computed from the formula $a^2 = b^2 + c^2 - 2\,bc \cos A$.

(*a*) Solve this formula for $\cos A$.

(*b*) Compute the number of degrees in angle A if $a = 5\sqrt{3}$, $b = 5$, $c = 10$.

(*c*) What change takes place in angle A as a increases while b and c remain constant?

Elimination of letters from formulas. Frequently it is a labor-saving device to eliminate a certain letter from two formulas before making substitutions.

Illustrative example. If $v = gt$ and $s = \frac{1}{2}gt^2$, complete the following table:

s	t	v
64	2	?
144	3	?
256	4	?
576	6	?

Analysis

First eliminate the letter g (because no value is assigned to g and we are not asked to solve for it) from the two equations, thus obtaining a formula containing the other letters but not g.

Solution

Solve one of the equations for g.

$$s = \tfrac{1}{2}gt^2.$$

$$2s = gt^2.$$

$$\frac{2s}{t^2} = g.$$

Now substituting this value of g in $v = gt$, we have

$$v = \frac{2s}{t^2} \cdot t.$$

$$v = \frac{2s}{t}.$$

Now the table can be completed by the use of this single formula.

Thus, when $s = 64$ and $t = 2$, $v = \frac{2(64)}{2} = 64.$

Find the other missing terms in the table.

EXERCISES

1. Eliminate a from $x = a + 2$ and $y = 3a - 5$ by solving for a in $x = a + 2$ and then substituting this value of a in $y = 3a - 5$.

2. (a) Eliminate h between $A = bh$ and $P = 2b + 2h$.
(b) Find the value of P when $A = 100$ and $b = 5$.

3. (a) Eliminate l between $V = lwh$ and $3l = 2w$.
(b) Find the value of V when $w = 3$ and $h = 7.5$.

4. Given the formulas $v^2 = 2gh$ and $v = gt$,
(a) Eliminate v and solve the resulting equation for h.
(b) Take $g = 32$ and find the value of h when $t = 2\frac{1}{2}$.

5. If $s = at - \frac{1}{2}gt^2$ and $V = gt$, find an expression not containing g.

6. Eliminate b from: $x = b + 1$, $y = b^2 - 1$.

7. Eliminate c from: $x = c - 2$, $y = c^3 - 3$.

8. Eliminate z between the two equations $x = z + 1$ and $y = 2z^2 + 1$.

9. Eliminate m from: $x = m^2 + 3$ and $y = m^2 - 1$, and find the value of y when $x = 5$.

10. If $\sin A = \frac{h}{b}$ and $\sin B = \frac{h}{a}$, form an equation not containing h.

11. Eliminate $\sin C$ from the equations $c = 2R \sin C$ and $T = \frac{1}{2}ab \sin C$.

12. Eliminate r from the formulas $y = r \sin \theta$ and $x = r \cos \theta$.

13. If $\sin^2 x - 4\cos^2 x + 2 = 0$ and $\sin^2 x + \cos^2 x = 1$, eliminate $\sin^2 x$ from the equations.

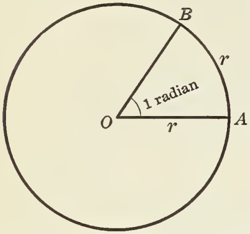
14. Eliminate $\sin C$ from $k_1 = \frac{1}{2}a_1b_1 \sin C$ and $k_2 = \frac{1}{2}a_2b_2 \sin C$.

15. From the two equations $r \cos \theta = a$ and $r \sin \theta = b$ derive an equation for r in terms of a and b only.

Radian Measure

You have been accustomed to measure angles in degrees in your work in mathematics. But there is another system, called *circular measure,* or *radian measure,* used in the theoretical treatment of the higher branches of mathematics.

The meaning of a radian. With center O and radius r describe a circle. Let arc AB equal radius r. Draw AO and BO. Then central angle AOB is called a radian.



A radian is an angle at the center of a circle which intercepts an arc equal in length to the radius.

The length of the circumference of a circle is equal to $2\pi r$. This means that the radius can be laid off on the circumference 2π times. There are therefore 2π radians about the center O. But since the number of degrees about the center O is 360, then 2π radians $= 360°$, or π radians $= 180°$.

$$\therefore\ 1 \text{ radian} = \frac{180°}{\pi} = \frac{180°}{3.1416} = 57.2957° = 57°\ 17'\ 45''$$

and $1 \text{ degree} = \dfrac{\pi}{180} \text{ radians} = \dfrac{3.1416}{180} \text{ radians} = .01745 \text{ radians.}$

Remember

To reduce a certain number of radians to degrees, multiply by $\dfrac{180}{\pi}$.

To reduce a certain number of degrees to radians, multiply by $\dfrac{\pi}{180}$.

Great care should be exercised when working with *radians,* because the word "radian" is frequently omitted from examples. For instance, we frequently write $\tan \frac{\pi}{2}$ and mean the tangent of $\frac{\pi}{2}$ radians or $\tan 90°$. While π is never equal to 180°, π (radians understood) does equal 180°.

Illustrative examples.

Example 1. Express 540° in radians.

Solution

$$540 \times \frac{\pi}{180} = 3\,\pi \text{ radians.}$$

Example 2. Express $\frac{7\,\pi}{6}$ radians as degrees.

Solution

$$\frac{7\,\cancel{\pi}}{\cancel{6}} \times \frac{\overset{30}{\cancel{180}}}{\cancel{\pi}} = 210°.$$

EXERCISES

Group A

1. Express each of the following angles in radians in terms of π.

(*a*) 180°.	(*f*) 30°.	(*k*) 150°.	(*p*) 250°.
(*b*) 90°.	(*g*) 15°.	(*l*) 175°.	(*q*) 270°.
(*c*) 60°.	(*h*) 40°.	(*m*) 190°.	(*r*) 300°.
(*d*) 120°.	(*i*) 360°.	(*n*) 210°.	(*s*) 200°.
(*e*) 45°.	(*j*) 135°.	(*o*) 240°.	(*t*) 22.5°.

2. The following angles are expressed in radians. Express each in degrees.

(*a*) $\frac{\pi}{2}$.	(*e*) $\frac{\pi}{6}$.	(*i*) $\frac{2\,\pi}{3}$.	(*m*) $\frac{\pi}{5}$.
(*b*) $\frac{\pi}{3}$.	(*f*) $\frac{\pi}{8}$.	(*j*) $\frac{3\,\pi}{4}$.	(*n*) $\frac{3\,\pi}{2}$.
(*c*) $\frac{\pi}{4}$.	(*g*) $\frac{\pi}{10}$.	(*k*) $\frac{4\,\pi}{5}$.	(*o*) $\frac{13\,\pi}{18}$.
(*d*) 2.	(*h*) 5.	(*l*) 1.5.	(*p*) $\frac{3}{4}$.

3. If the radius of a circle is 6 inches, find the number of radians in the central angle which intercepts an arc of (*a*) 2 inches; (*b*) 5 inches; (*c*) 3.5 inches.

Group B

4. Express the following in radians.

(*a*) 25° 30′.	(*e*) 34° 9′.	(*i*) 11.25°.
(*b*) 15° 15′.	(*f*) 23° 45′.	(*j*) 7.75°.
(*c*) 18° 45′.	(*g*) 17.2°.	(*k*) 3.3°.
(*d*) 28° 27′.	(*h*) 28.75°.	(*l*) 8.2°.

Group C

5. Find the radius of a circle in which an arc of length 35 inches subtends an angle of 3.5 radians.

6. The bob of a pendulum 10 inches long swings through an arc of 4 inches. Through how many radians does the pendulum swing? How many degrees?

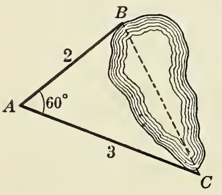
7. Find the length of an arc of 50° on a circle of radius $2\frac{1}{2}$ feet.

8. Through how many radians does the hour hand of a clock turn in 45 minutes?

9. Prove that the area of a sector of a circle is $\frac{r^2\theta}{2}$ (r being the radius, and θ the angle at the center, in radians).

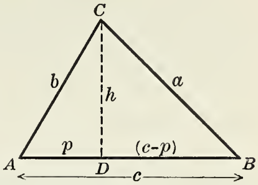
The Law of Cosines

Problem. In order to find the distance between two points B and C separated by a lake, station A was chosen and by measurement AB was found to be 2 miles, AC 3 miles, and angle $BAC = 60°$. Find the distance from B to C.



We have here a new situation, because up to the present we have studied triangles in which one of the angles was a right angle. But in this triangle we do not know that one of the angles is a right angle. If we knew a certain relationship between the sides and one angle of any triangle, we could solve this problem.

Let us consider acute triangle ABC in which the altitude CD is drawn. Using the Pythagorean Theorem we have,



$a^2 = h^2 + (c - p)^2.$

$a^2 = h^2 + c^2 - 2\,cp + p^2.$

$a^2 = (h^2 + p^2) + c^2 - 2\,cp.$ (1)

But $h^2 + p^2 = b^2$ (2)

and

$\frac{p}{b} = \cos A$; i.e., $p = b \cos A.$ (3)

Substituting (2) and (3) in (1), we get $a^2 = b^2 + c^2 - 2\,bc \cos A.$

Similarly, $b^2 = a^2 + c^2 - 2\,ac \cos B$

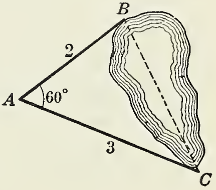
and $c^2 = a^2 + b^2 - 2\,ab \cos C.$

This relation, expressing one side of a triangle in terms of the other two sides and their included angle, is called the

Law of Cosines

$$\boldsymbol{a^2 = b^2 + c^2 - 2\,bc \cos A.}$$
$$\boldsymbol{b^2 = a^2 + c^2 - 2\,ac \cos B.}$$
$$\boldsymbol{c^2 = a^2 + b^2 - 2\,ab \cos C.}$$

Now let us return to the problem at the beginning of this discussion. We are given AB which may be called $c = 2$ miles, AC which may be called $b = 3$ miles, and angle $A = 60°$. We are required to find BC, or a.



Solution

$a^2 = b^2 + c^2 - 2\,bc \cos A.$

Substituting $a^2 = 9 + 4 - 12\,(\frac{1}{2})$

$= 7.$

$\therefore\ a = \sqrt{7} = 2.6$ miles correct to the nearest tenth.

Check

1. How could you calculate the altitude to side b?

2. How could you then calculate the segments of side b made by the altitude to that side?

3. With the values of (1) and (2) just obtained how would you find side a?

4. Can this procedure be used to check the above example?

Illustrative example. Find the number of degrees in the smallest angle of the triangle whose sides are 7, 9, and 10.

Analysis

In this triangle let $a = 7$, $b = 9$, $c = 10$. Since the smallest angle in a triangle lies opposite the smallest side, we are to find angle A.

Solution

Substituting in the law of cosines formula,

$$a^2 = b^2 + c^2 - 2\,bc \cos A, \text{ we have}$$
$$49 = 81 + 100 - 180 \cos A.$$
$$\cos A = \tfrac{132}{180} = .7333.$$
$$A = 43°.$$

EXERCISES

Group A

1. Given $b = 30$, $a = 50$, $C = 30°$. Find c.

2. Given $a = 5$, $b = 5$, $C = 60°$. Find c.

3. Given $a = 15$, $b = 15$, $C = 45°$. Find c.

4. Given $a = 16$, $c = 10$, $B = 30°$. Find b.

5. Given $b = 5$, $c = 6$, $A = 35°$. Find a.

6. Find the number of degrees in the smallest angle of the triangle whose sides are:

(*a*) 6, 8, 13. (*c*) 4, 8, 10. (*e*) 6, 7, 8.
(*b*) 11, 10, 9. (*d*) 5, 12, 13. (*f*) 3, 7, 6.

Group B

7. Solve the equation $a^2 = b^2 + c^2 - 2\,bc \cos A$ for $\cos A$.

8. Solve the equation $b^2 = a^2 + c^2 - 2\,ac \cos B$ for $\cos B$.

9. The sides of a triangle are 3, 4, and 5. Find the number of degrees in the largest angle.

10. The sides of a triangle are 5, 6, and 7. Find the smallest angle.

11. Given the sides of a triangle 35, 40, and 50, find the three angles of the triangle. (Check by adding the angles.)

12. The sides of a triangle are 40, 50, and 60. Find the three angles correct to the nearest degree.

Group C

13. To find the distance between two points A and C which are separated by a swamp, a surveyor measures inland a line from A to B of length 120 yards, then from B to C of length 100 yards. If the angle ABC is 60°, how far is it from A to C?

14. Find the perimeter of a triangle where $a = 20$ feet, $b = 30$ feet, and angle C is 85°.

15. If $\dfrac{\cos A}{b} = \dfrac{\cos B}{a}$, prove that the triangle ABC is either isosceles or right.

16. The angle between the diagonals of a parallelogram is 40° and the diagonals are 6 and 8 feet in length. Find the length of the shorter side.

Something to think about.

1. If one of the angles of a triangle is 90°, what does the formula for the law of cosines become? Do you recognize this result?

2. Do you think the Pythagorean theorem relations are special cases of the law of cosines?

Problems

The ability to express relations, form an equation, and solve the equation is the chief value derived from studying algebra, because it will train you to think, to analyze situations, and to solve the general problems that come up in your life.

NUMBER PROBLEMS

Group A

1. Separate 150 into two parts such that one part is twice as large as the other.

2. Two numbers differ by 2, and their sum is 38. Find the numbers.

3. Find three consecutive even numbers whose sum is 66.

4. Find four consecutive odd numbers such that the last is three times the first.

5. The sum of the three sides of a triangle is 31 feet. The second exceeds the first by 3 feet and the third is twice the first. Find the sides of the triangle.

6. Find a number whose third part exceeds its fourth part by 2.

7. Find two consecutive numbers such that $\frac{1}{5}$ of the smaller exceeds $\frac{1}{8}$ of the larger by 1.

8. A boy's grades in three subjects are 84, 87, and 93. What must his grade in a fourth subject be to make his general average 90?

9. What number must be added to the numerator of $\frac{14}{25}$ to make the value of the resulting fraction $\frac{3}{5}$?

Group B

10. Divide 56 into two parts such that the greater divided by the less shall have a quotient of 3 and a remainder of 4.

11. The sum of two numbers is 72; if the greater is divided by the smaller, the quotient will be 2 and the remainder 6. Find the numbers.

12. By what number must 72 be divided so that the quotient will be 4 and the remainder 12?

13. A man sold 2 acres more than $\frac{3}{5}$ of his farm and had 4 acres less than half of it left. Find the number of acres in the farm.

14. When a certain number of shares of stock are sold at \$20 each, the same amount is realized as when all but 15 of the shares are sold for \$22 each. How many shares are there?

Group C

15. A broker bought a certain number of shares of stock for \$1200. After the price of each share had advanced \$40 he sold all but three of his shares for \$1440. How many shares did he buy?

16. In an orchard containing 2800 trees, the number of trees in each row is 10 less than twice the number of rows. How many trees in each row?

17. The admission tickets for an entertainment were 25¢ each for adults and 10¢ each for children; the turnstile shows that 385 persons entered and the gate receipts were \$62.65. How many children entered?

18. The admission to an entertainment was 50¢ for adults and 25¢ for children; the total receipts from 150 tickets were \$62.50. Find the number of adults admitted.

19. The difference between two numbers is 8, and twice the sum of their reciprocals is equal to 3 times the difference of their reciprocals. Find the numbers.

20. The ratio of two numbers is 3 : 2. If the larger number is reduced by 5 and the smaller increased by 10, their ratio is 1. Find the numbers.

AGE PROBLEMS

Group A

1. If x represents John's present age and y represents Mary's present age, express algebraically:

(*a*) John's age 3 years from now; m years from now.

(*b*) Mary's age 5 years from now; n years from now.

(*c*) John's age 5 years ago; s years ago.

(*d*) Mary's age 3 years ago; t years ago.

(*e*) The sum of John's and Mary's present ages.

(*f*) The sum of John's age 3 years ago and Mary's age 5 years ago.

(*g*) The sum of their ages 3 years ago; d years ago.

(*h*) The sum of their ages in 10 years; in h years.

(*i*) The difference of their present ages.

(*j*) 6 years ago John was twice as old as Mary.

Illustrative example. The sum of the present ages of John and his father is 65. In 10 years John will be the same age that his father was 25 years ago. Find their present ages.

Solution

Let x represent John's present age.

Name	Present Age	Past Age	Future Age
John	x		$(x + 10)$
Father	$(65 - x)$	$(40 - x)$	

$$\therefore\ (x + 10) = (40 - x).$$
$$x = 15.$$

John is 15 years old and his father is 50. Check in the original problem.

2. The age of the elder of two boys is 3 times that of the younger; 4 years ago it was 5 times that of the younger. Find the age of each.

3. In 10 years a man will be 3 times as old as he was 15 years ago. Find his present age.

4. A man is 3 times as old as his son, and his daughter is 2 years younger than the son. If the sum of their ages is 48 years, find the age of the father.

5. Bernard is $\frac{1}{3}$ as old as his father. In 8 years he will be $\frac{5}{11}$ as old as his father. How old is each?

6. Jane's age is $\frac{1}{3}$ of her father's age. In 15 years she will be $\frac{1}{2}$ as old as her father. How old is Jane?

7. A man 40 years old has a daughter Catharine 10 years old. In how many years will the father be 3 times as old as Catharine?

Group B

8. A father is now 48 years old and his daughter Ruth is 16 years old. How many years ago was the age of the father 5 times the age of his daughter?

9. Norma's age is to Morton's age as 2 : 3 and the difference of their ages is 6. Find their ages.

10. A is 36 years old and B is $\frac{2}{3}$ as old. How many years ago was B one-half as old as A was then?

11. Seymour is 5 years older than Gerald; Gerald is 4 years older than Herbert; Herbert is 3 years older than Michael. In 3 years the sum of their ages will be 74 years. Find their ages.

12. Eighteen years ago a man was $\frac{2}{5}$ as old as he will be in 12 years. What is his present age?

Group C

13. A is 15 years older than B, and A's age is as much above 20 as B's is below 45. Find their ages.

14. A's age is twice C's age and B's age exceeds C's age by 5. 20 years ago A's age was three times B's age. Find the age of each of the three men.

15. The age of the older of two sons exceeds the age of the younger by 2 years. The sum of their present ages is 14 years less than their father's age. Ten years ago the father's age was 3 times the sum of the ages of his two sons. Find their ages.

16. In 2 years A will be 3 times as old as B and 8 years ago A was 5 times as old as B. Find their present ages.

UNIFORM MOTION PROBLEMS

Group A

1. (*a*) Explain what each letter represents in the formula $d = rt$. Solve this formula for r; for t.

(*b*) A man walks 2 miles per hour. How far will he walk in 3 hours? x hours? $(x + 1)$ hours? $(x - 2)$ hours?

2. A travels r miles per hour. B travels 5 miles per hour faster. What is B's rate? How far does A travel in 3 hours? How far does B travel in 3 hours? What is the sum of the distances traveled by A and B in 3 hours?

3. An automobile travels at the rate of x miles an hour. How long will it take to travel 250 miles? m miles? If it increases its speed 10 miles an hour, will the time taken to travel the above distances increase or decrease?

4. A man has 5 hours at his disposal for travel. At what rate must he travel to cover 50 miles? y miles? $(y - 3)$ miles? If he wishes to cover the above distances in 3 hours, would he increase or decrease his rate?

5. A man's rate of rowing in still water is 2 miles an hour. If he rows on a stream which flows 1 mile an hour, what is his rate with the current (downstream)? against the current (upstream)?

6. An airplane can go 125 miles an hour in still air. How fast can it travel against a wind blowing x miles an hour? with a wind blowing y miles an hour?

Illustrative example. A is 100 miles from B. An automobile at A starts for B at the rate of 25 miles an hour at the same time that an automobile at B starts for A at the rate of 30 miles an hour. In how many hours will they meet?

Solution. Let x = the number of hours.

NAME	r ×	t =	d
A	25	x	$25x$
B	30	x	$30x$
		Total	100

$$25x + 30x = 100.$$
$$55x = 100.$$
$$x = 1\tfrac{9}{11} \text{ hours.}$$

Check

$$\text{Distance A goes} + \text{distance B goes} = 100.$$
$$25 \times 1\tfrac{9}{11} + 30 \times 1\tfrac{9}{11} = 100.$$
$$100 = 100.$$

7. John and Robert start toward each other at the same time from points 300 miles apart. If John travels 30 miles an hour, how fast must Robert travel that they may meet in 6 hours?

8. A passenger and a freight train start toward each other at the same time from points 420 miles apart. If the rate of the passenger exceeds that of the freight by 20 miles an hour, and they meet after 7 hours, what must be the rate of each?

9. Two airplanes start from Denver, one traveling east, the other west, at speeds differing by 20 miles an hour. They are 500 miles apart in $2\frac{1}{2}$ hours. What are their speeds?

10. A Boy Scout group is traveling 3 miles an hour. At what rate must a boy who started one hour late walk to overtake them in 3 hours?

11. One hour after a messenger, traveling 30 miles an hour, had left an army post, it was decided to cancel the message. How fast must the second messenger travel to overtake the first in 5 hours?

12. A train leaves Detroit for the east at 9 A.M., traveling 30 miles an hour. At 11 A.M. a fast express follows from Detroit on the same track. At what rate must the express run so that the dispatcher can arrange to have it pass the first train at 2 P.M.?

13. Two motor boats leave each other traveling in opposite directions at speeds in the ratio 3 : 4, and in 4 hours are 182 miles apart. Find their rates.

14. Two automobiles traveling in opposite directions start from the same place at the same time. The ratio of their rates is 7 : 9. After $3\frac{1}{2}$ hours one has traveled 35 miles more than the other. Find their rates.

Group B

15. A and B start from the same town and move in opposite directions. A's rate is 30 miles an hour and B's 25 miles an hour; B starts 2 hours later than A. In how many hours after A starts will they be 280 miles apart?

16. The distance between two towns is 120 miles. Two automobiles start at the same time toward each other from these towns at rates in the ratio 2 : 3. What are their rates if they meet in 3 hours?

17. A messenger was sent at 15 miles an hour to army headquarters 75 miles away. Two hours later an automobile was dispatched at 40 miles an hour to recall him. Will the automobile overtake him before reaching headquarters, and if so, how far from headquarters?

18. A certain city official missed a train by 2 hours and hired an airplane to catch it. The airplane went 40 miles an hour faster than the train and caught it in 2 hours. What must have been the rate of the train?

19. A messenger going 5 miles an hour was sent to an army 50 miles away. Five hours later an automobile party leaves the army at 20 miles an hour to meet him. At what distance from the army should they expect to meet him?

20. A and B set out at the same time to meet, A traveling 12 miles an hour and B 10 miles an hour. When they meet, A has passed 25 miles beyond the half-way point. How far apart were they at first?

Group C

21. Two trains travel in opposite directions starting from the same place at the same time at rates of 23 and 34 miles an hour. After traveling for 2 hours, the slower train increases its rate to 25 miles an hour. After how many hours will they be 299 miles apart?

22. Two trains are 1060 miles apart. They travel toward each other, A at 50 miles an hour and B at 30 miles an hour. If B starts 2 hours later than A, how far from each terminal must the train dispatcher arrange to have them pass?

23. A train dispatcher needs a formula for arranging the time for trains to pass. Make him one from the following data. A train traveling m miles an hour starts t hours after another running r miles an hour. In how many hours will the later train overtake the earlier?

24. Two trains are to start at the same time from towns 120 miles apart and travel toward each other, one going 10 miles an hour faster than the other. If the faster train is to make a half hour stop on the way, at what rates must the train dispatcher send them that they may pass at a city midway between the terminal points?

25. The traffic manager desires to have one train leave Portland at 8 A.M., and another, running 30 miles an hour, to leave 40 minutes later and overtake the first train at 11 : 20 A.M. At what rate must he arrange to have the engineer run the first train?

26. A, B, and C start from the same place at 17, 19, and 21 miles an hour respectively. B starts 3 hours after A. How long after B starts must C leave in order to overtake A at the same time that B overtakes him, and how far will each have traveled by that time?

ROUND TRIP PROBLEMS

Illustrative example. Charles has just two hours spare time. How far may he ride in a bus which travels at 10 miles an hour, so as to return home in time, walking back at the rate of 4 miles an hour?

Solution

Let x = the distance.

NAME	r	$\times \; t\left(=\frac{d}{r}\right)$	$= d$
Going	10	$\frac{x}{10}$	x
Returning	4	$\frac{x}{4}$	x
	Total	2	

$$\frac{x}{10} + \frac{x}{4} = 2.$$

$$2x + 5x = 40.$$

$$x = 5\tfrac{5}{7} \text{ miles.}$$

Check in the original problem.

Group A

1. A man must wait 2 hours for a connecting train. To pass the time, he wishes to ride out into the country on a street car at 10 miles an hour and walk back at 3 miles an hour. How far should he ride so that he can be back in time for his train?

2. Katherine has 6 hours at her disposal. How far can she ride at 9 miles an hour so as to return on time, walking back at 3 miles an hour?

3. A boy left home on his bicycle at 12 miles an hour. After riding a certain distance, he punctured a tire and had to walk home at 4 miles an hour. He was away from home 5 hours. How far did he ride?

4. Henry had 4 hours to wait for a train, so he took a walk at 4 miles an hour and returned by trolley at 9 miles an hour, arriving just in time for his train. How far did he walk?

5. John walked to the park at $2\frac{1}{2}$ miles an hour. He stayed there 2 hours, then returned home at the rate of 3 miles an hour. If he was out 8 hours, how far from home was the park?

Group B

6. Ruth wishes to devote 4 hours and 15 minutes to a trolley ride and the walk back home. If the trolley goes 10 miles an hour and she walks 4 miles an hour, how far should she go?

7. Mr. Fox set out at 7 A.M. and drove to a certain town at 10 miles an hour. He returned at 4 miles an hour and reached home at 7 : 30 P.M., having lunched one hour and spent another hour transacting his business. How far away was the town?

8. A Scout Troop set out on a 9-mile hike. Their rate returning was one-half the rate going and the time for the round trip was 9 hours. Find the rates going and returning.

9. A man drove to a town 12 miles away. If the ratio of his rates going and returning was 3 : 2 and he arrived home 4 hours after he started, what was his rate going to the town?

Group C

10. An automobilist goes to a place 72 miles distant and then returns, the round trip occupying 9 hours. His speed returning is 12 miles an hour faster than his speed in going. Find his rate going and his rate returning.

11. An automobile travels 100 miles and then returns, the round trip taking 9 hours. If the rate of speed returning is 5 miles an hour less than the rate of speed going, find the rate each way.

12. A man travels 80 miles and returns in 9 hours. If the rate returning was 4 miles an hour faster than the rate going, find the rate each way.

13. A man travels to a certain place by automobile and returns on foot. If his rate going is a miles an hour and rate returning is b miles an hour, what is the distance he travels by automobile if he wishes to make the round trip in h hours?

COIN PROBLEMS

Illustrative example. A bellboy has coins consisting of nickels, dimes, and quarters, having a total value of $3.70. If there are 4 more dimes than nickels and twice as many quarters as dimes, how many coins of each kind has he?

Solution

Let x = number of nickels.

Name	Number	Value of each coin in pennies	Total value in pennies
Nickels	x	5	$5x$
Dimes	$(x + 4)$	10	$10(x + 4)$
Quarters	$2(x + 4)$	25	$50(x + 4)$
		Total amount in pennies	370

$$
\begin{aligned}
5x + 10(x + 4) + 50(x + 4) &= 370. \\
5x + 10x + 40 + 50x + 200 &= 370. \\
65x &= 130. \\
x &= 2 \text{ nickels.} \\
x + 4 &= 6 \text{ dimes.} \\
2(x + 4) &= 12 \text{ quarters.}
\end{aligned}
$$

Check in the original problem.

Group A

1. A merchant has \$1.25 in nickels and dimes. There are twice as many dimes as nickels. How many of each kind has he?

2. \$9.50 is made up of quarters and half-dollars. The number of half-dollars exceeds the number of quarters by 4. Find the number of coins of each denomination.

3. In a money box there are 5 more dimes than half-dollars. The total value of all the coins is \$2.90. How many of each coin are there?

4. A boy's bank contains dimes and quarters, having a total value of \$1.60. If the number of dimes is 3 more than four times the number of quarters, how many of each kind of coin does the bank contain?

5. In exchange for a \$2 bill, a boy received dimes, nickels, and quarters. If he received the same number of each denomination, how many of each did he receive?

6. A man has \$2.96 in dollars, dimes, and cents. He has $\frac{2}{3}$ as many cents as dimes and $\frac{1}{3}$ as many dollars as cents. How many coins of each kind has he?

Group B

7. A collection of 25 coins consisting of dimes and quarters amounts to \$4.75. How many coins of each kind are there?

8. A collection of 17 coins, consisting of nickels and dimes, amounts to \$1.45. How many coins of each kind are there?

9. A purse has 19 coins, some of which are quarters and the remainder dimes. If the coins are worth \$4.15 all together, how many are there of each kind?

10. A boy has \$4.00 in half-dollars and nickels, there being 35 coins in all. How many coins of each kind has he?

11. \$6.05, in dimes and quarters, were distributed among 50 boys. If each boy received one coin, how many dimes and quarters were there?

Group C

12. Five dollars, in dimes and half-dollars, were distributed among 30 boys. If each received one coin, how many received dimes?

13. I have a sum of money amounting to \$6.00 consisting of nickels, dimes, quarters, and half-dollars. I have twice as many dimes as nickels, as many quarters as dimes and nickels together, and $\frac{1}{3}$ as many half-dollars as quarters. How many of each kind of coin have I?

14. 20 coins amounting in value to \$2.60 consist of nickels, dimes, and quarters. The number of nickels is $\frac{3}{4}$ the number of dimes. How many of each coin are there?

15. A purse contains a certain number of nickels and dimes having a total value of d dollars. If there are c coins all together, how many of each kind are there?

WORK PROBLEMS

Group A

1. A man can do a piece of work in 5 days. What fractional part of it can he do in 1 day? 2 days? 3 days? x days? m days?

2. A man painted his barn in d days. What part of the work could he do in 1 day? 2 days? 4 days? y days? g days?

3. A man can address 72 dozen envelopes in $2\frac{1}{2}$ days. What part of the work can he do in 1 day? 2 days? 5 days? x days?

Illustrative example. If A can do a piece of work alone in 4 days and B can do it alone in 2 days, how long will it take the two working together to complete the job?

Solution

Let x = the number of days taken to complete the job working together.

NAME	CAN DO WORK ALONE IN	CAN DO ALONE IN 1 DAY	CAN DO ALONE IN x DAYS
A	4	$\frac{1}{4}$	$\frac{x}{4}$
B	2	$\frac{1}{2}$	$\frac{x}{2}$

= Whole job

$$\therefore \frac{x}{4} + \frac{x}{2} = 1.$$

$$x + 2x = 4.$$

$$x = 1\tfrac{1}{3} \text{ days.}$$

Check in the original problem.

4. It takes A 3 times as long as it takes B to perform a certain task alone. Working together they could finish the task in 4 days. How long does each take to do the task alone?

5. Mr. Smith could paint his house alone in 14 days. Mr. Jones could paint it alone in half the time. How long would it take the two working together to do the job?

6. A can do a piece of work alone in one-fourth the time it takes B. How long would it take each working alone to do the job if working together they complete the job in 4 days?

7. A can do a job alone in 3 days. A and B working together can do it in 2 days. How long would it take B alone to do the job?

8. A can do a job alone in 4 days. How long would it take B alone to do the job if working together they could finish it in 3 days?

Group B

9. A can do a piece of work in 3 days, B in 4 days, and C in 6 days. How long will it take them to do it working together?

10. A can do a job alone in 4 days. After he has worked 1 day B joins him, and the two finish the job in 2 days. How long would it take B to do the job alone?

11. A can do a piece of work alone in 12 days and B can do the same work alone in 8 days. After A works alone for two days, he hires B. In how many days can the work be completed with B's assistance?

12. A cistern can be filled by two pipes in 5 hours and 8 hours respectively, and can be emptied by a third in 10 hours. In what time will the cistern be filled, if all three pipes are running together?

Group C

13. A can do a piece of work in 10 days and B the same work in 15 days; after they have worked together 5 days, B finishes the work. How long did it take B to finish the work?

14. Two men together can do a piece of work in 2 days. How long will it take each to do the work alone, if it takes one 3 days longer than the other?

15. Two men A and B can dig a ditch in 20 days; it would take A 9 days longer to dig it alone than it would take B. How long would it take B alone?

16. A tank can be filled by two pipes in 30 minutes and 20 minutes respectively, and can be emptied by a third in 50 minutes. In what time will the tank be filled, if all pipes are running together?

INVESTMENT, INTEREST, AND BUSINESS PROBLEMS

Illustrative example. A man invested \$1200 in two enterprises, one paying 5% and the other 4%. If his annual income from both investments is \$56, how much did he invest in each?

Solution

Let x = amount invested at 5%.

NAME	p $\times$	r $=$	I
First	x	.05	$.05\,x$
Second	$(1200 - x)$	.04	$.04(1200 - x)$
		Total	56

$$
\begin{aligned}
.05\,x + .04(1200 - x) &= 56.\\
5\,x + 4(1200 - x) &= 5600.\\
x &= 800.\\
1200 - x &= 400.
\end{aligned}
$$

He invested \$800 at 5% and \$400 at 4%.

Check

$$
\begin{aligned}
\$800 \text{ @ } 5\% &= \$40.\\
\$400 \text{ @ } 4\% &= \underline{\$16.}\\
\text{Total} \quad & \quad \$56.
\end{aligned}
$$

Group A

1. Mr. Briggs has \$20,000 to invest. He prefers a mortgage paying 5% interest, but since he needs an income of \$1200 a year, he must invest part in stock paying 9%. How much must he invest in stock?

2. Henry Stone has \$43,000 to invest, part at 6% and part at 7%. If he must have an income of \$2790 a year, how much should he invest at each rate?

3. George Wynn has \$9000 in a savings bank which pays 4% annually. He wishes to increase his income to \$480 a year by investing part of this money at 7%. How much must he invest at 7%?

4. A man invests $\frac{4}{7}$ of his capital at 5% and the rest of it at 4%. If his annual income is \$1600, what is his capital?

5. A man invests $\frac{1}{4}$ of his capital at 5%, $\frac{2}{5}$ of it at 4%, $\frac{1}{6}$ of it at 3%, and the rest of it at 2%. If his total annual income is \$892, what is the capital?

6. A sum of \$5000 is invested partly at 3% and partly at 5%. The interest on the first investment is \$30 more a year than the interest on the second investment. How much was invested at each rate?

7. A dealer bought a number of sheep for \$700; after 10 had died he sold the remainder at a profit of \$2 each, thereby realizing a profit of \$20 on the whole transaction. How many sheep did he buy?

8. A man bought a number of books for \$120. When the price advanced \$2 on each book, he sold all the books but one for \$132. How many books did he buy?

9. A merchant bought a number of barrels of apples for \$136. He retained 3 barrels and sold the remainder at an advance of \$3 per barrel, thereby gaining \$18. How many barrels did he buy?

10. A merchant bought some pieces of cloth for \$210. If he had bought 5 more pieces for the same money, he would have paid \$1 less for each piece. Find the number of pieces purchased.

11. A butcher buys a certain number of pounds of meat for \$9. If he had paid 5 cents more a pound, he would have received 2 pounds less for his money. How many pounds did he buy?

12. If a dealer gains a cents on the cost of an article and the article costs b dollars, what per cent does he gain?

Group B

13. A man bought an article for x dollars and sold it for \$24, thus gaining x per cent. What was the cost?

14. A man sold some goods for \$22 and his gain per cent was equal to one-half the cost in dollars. What was the cost of the goods?

15. A man wishes to invest \$12,000, part at 4% and the remainder at 7%, so as to average 6% on his money. How much should he invest at each rate?

16. A man has a certain sum of money to invest. He finds that by buying 5 per cent stock at 90, his income will be \$30 more per annum than if he bought 8 per cent stock at 150. How much money has he to invest?

Group C

17. Goods cost a merchant \$72. At what price should he mark them so that he may sell them at a discount of 10% from this marked price and still make a profit of 20% of the sale price?

18. A certain number of bolts can be bought for a dollar. If 10 more could be bought for the same money, the price would be half a cent less per dozen. What is the price per dozen?

19. A man invests \$2720 in railroad stock, a part at 95 yielding 2% and the balance at 82 yielding 3%; his income from both investments is \$70. Find the amount invested in each kind of stock.

SCIENCE PROBLEMS — MIXTURES AND SOLUTIONS

Remember that the fraction of any ingredient in a mixture is equal to the total amount of the ingredient divided by the total mixture.

Group A

1. A solution contains 4 ounces of salt and 16 ounces of water.

(*a*) What fraction of the solution is salt? What per cent?

(*b*) What fraction of the solution is water? What per cent?

(*c*) If 5 ounces of salt are added to the original solution, what fraction of the new solution is salt? What per cent? By adding salt to the original solution does it increase or decrease the percentage of salt in the solution?

(*d*) If 5 ounces of water are added to the original solution, what fraction of the new solution is salt? What per cent? By adding water to the original solution does it increase or decrease the percentage of the salt in the solution?

(*e*) What is meant by "diluting" a solution?

2. 40 ounces of a solution of soda in water is 10% soda.

(*a*) How many ounces of soda are in the solution? How many ounces of water?

(*b*) What fraction of the solution is soda? What per cent? What fraction of the solution is water? What per cent?

(*c*) 8 ounces of soda are added to the original solution. What is the total amount of soda in the new solution? What is the total amount of the new solution? What fraction of the new solution is soda? What per cent?

Illustrative example. Six quarts of gasoline are mixed with 8 quarts of kerosene. How many more quarts of kerosene must be added to make the mixture $\frac{3}{4}$ kerosene?

Solution

Let x = number of quarts of kerosene added.

Content	Original Solution	Changed Solution
Kerosene	8	$8 + x$
Gasoline	6	6
Solution	14	$14 + x$

$$\frac{8+x}{14+x}=\frac{3}{4}.$$
$$4(8+x)=3(14+x).$$
$$32+4x=42+3x.$$
$$x=10.$$

Ten quarts of kerosene must be added.

Check in the original problem.

3. In a 28-ounce solution of salt in water, there are 3 ounces of salt. How many ounces of salt must be added to make a new solution that will be $\frac{1}{6}$ salt?

Illustrative example. A mixture contains 10 gallons of acid and 25 gallons of water. How many gallons of water must I add to dilute the mixture so that it will be a 10% solution of acid?

Solution

Let x = number of gallons of water added.

Content	Original Solution	Changed Solution
Water	25	$25+x$
Acid	10	10
Solution	35	$35+x$

$$10\% \text{ of } (35+x)=10.$$
$$.10(35+x)=10.$$
$$35+x=100.$$
$$x=65.$$

Check in the original problem.

4. A solution contains 4 ounces of borax and 16 ounces of water. How many ounces of water must be added to reduce it to a 10% solution of borax?

5. A solution of 60 gallons of acid is 25% acid. How many gallons of acid must be added to make the solution a 40% solution?

6. How much water must be added to 50 pounds of a 10% solution of salt to reduce it to a 5% solution?

7. How much water must be added to a pint of 20% solution of argyrol to reduce to a 10% solution?

Group B

8. How many ounces of alloy must be added to 30 ounces of silver to make a composition of 60% silver?

9. A goldsmith has two alloys of gold, the first being $\frac{3}{4}$ pure gold, the second $\frac{5}{12}$ pure gold. How much of each must be taken to produce 50 ounces of an alloy which will be $\frac{2}{3}$ pure gold?

10. A solution contains b ounces of borax and w ounces of water. How many ounces of water must be added to make a c per cent solution?

11. In m ounces of a solution containing salt and water there are n ounces of salt. How many ounces of salt must be added to make a new solution that will be p per cent salt?

Group C

12. A radiator contains 20 quarts of a 20% mixture of alcohol and water. How much of this mixture must be drained off and replaced by alcohol so that the result will be a 36% mixture?

13. An automobile radiator has a capacity of 20 quarts. It is filled with a mixture of water and alcohol of which 10% is alcohol. How much of the mixture must be drawn off and replaced with pure alcohol so that the radiator will contain a mixture that is 25% alcohol?

14. An alloy of 20 lb. contains copper and tin in the ratio of 4 : 1. Another alloy contains copper and tin in the ratio of 3 : 1. How many pounds of the second must be added so that the resulting alloy contains copper and tin in the ratio of 10 : 3?

15. How many pounds of a 4% solution of salt must be added to 24 pounds of a 12% solution of salt to obtain a 10% solution of salt?

The Use of Equations

In the Ahmes papyrus (about 1650 B.C.) already referred to, we find equations solved for the first time. The unknown quantity is called "hau" or heap, and one of the problems reads "heap, its $\frac{1}{7}$, its whole, it makes 19," which in our present-day symbols would be written $x + \frac{1}{7}x = 19$. However, it was not until the seventeenth century that Descartes began the use of the letters x, y, and z for unknown quantities.

CUMULATIVE REVIEW

Chapters III, IV, and V

1. Which of these statements are true? Which are false?

(*a*) $x = 0°$ is a root of the equation $\tan^2 x - \tan x = 0$.

(*b*) Any number and its reciprocal may be considered as the factors of 1.

(*c*) $\dfrac{b-a}{a-b} = -1$.

(*d*) $(2a + \frac{1}{2}) \div (2a + \frac{1}{2}) = 0$.

(*e*) In the equation $y = x + \dfrac{1}{x}$ if x is greater than 1 and increasing, then y is positive and increasing.

(*f*) -1.3 is a root of the equation $3x + 4 = 0$.

(*g*) A fraction can be a root of a fractional equation.

(*h*) $\dfrac{c-b}{a}$ is a root of the equation $ax + b = c$.

(*i*) In the formula $C = \frac{5}{9}(F - 32)$, C can never have a larger value than F if F is positive.

(*j*) If b and c are constant in the formula $P = bc \sin A$, the maximum value of P is obtained when $A = 90°$.

2. Complete each of the following statements:

(*a*) If 5, $a^2 - b^2$, and $a^2 + b^2$ are factors of a certain expression, then all the prime factors of that expression are ? .

(*b*) The number of prime factors in the expression $x^4 - 81$ is ? .

(*c*) If $\frac{1}{\tan x}$ is divided by its reciprocal, the answer is ? .

(*d*) If the perimeter of an equilateral triangle is expressed by the fraction $\frac{9x + 6y}{x + 2y}$, each side may be represented by the fraction ? .

(*e*) If $\sin^2 A = \frac{1}{4}$, $\cos^2 A =$? .

(*f*) $\frac{5}{6}\pi$ radians = ? degrees.

(*g*) If n is an odd number, the next larger odd number is ? and the next smaller even number to n is ? .

(*h*) If $2x$ and $5x$ represent the present ages of A and B, respectively, the ratio of their ages 5 years hence may be represented as ? .

(*i*) Between 2 P.M. and 5 P.M. a ferry boat plying at a uniform rate covers a distance of s miles. At the same rate it can cover ? miles in t hours.

(*j*) If a boy has q quarters and d dimes and spends one-fifth of his money, he has ? cents left.

3. By what binomial must $2a^x - b^y$ be multiplied to yield as a result the trinomial $6a^{2x} + 7a^xb^y - 5b^{2y}$?

4. (*a*) Factor: $x^3 - 64x$.

(*b*) For positive values of x, which factor will always have the largest value?

(*c*) For positive values of x, which factor will always have the least value?

5. Simplify: $$\frac{\dfrac{\cos A}{\sin A} + \dfrac{\sin A}{\cos A}}{\dfrac{\cos A}{\sin A} - \dfrac{\sin A}{\cos A}}.$$

6. Prove the identity: $\tan x - \cot x = \dfrac{1 - \cot^2 x}{\cot x}$.

7. The sides of a triangle are $\frac{a}{a+b}$, $\frac{-b}{b-a}$, and $\frac{2ab}{a^2-b^2}$. Find the perimeter of the triangle.

8. Solve and check:

$$(2 \sin x - 1)(3 \sin x - 1) = (\sin x - 5)(6 \sin x + 5) + 36.$$

9. If the product of three consecutive numbers is divided by each of the numbers in turn, the sum of the three quotients thus obtained diminished by three times the square of the first number is 50. Find the numbers.

10. A man has \$6000 invested at $4\frac{1}{2}\%$. How much more must he invest at 6% in order to realize 5% on his total investment?

11. One pipe can fill a tank in 30 minutes while another can fill the same tank in 20 minutes. After the first pipe has poured water into the tank for 15 minutes, the second pipe is turned on. How long will it take both pipes to complete filling the tank?

12. As a reward for saving a boy, a lifesaver was given a purse containing \$11 in dimes, half-dollars, and quarters. If the number of half dollars is 4 less than twice the number of dimes, and the number of quarters is half the number of half-dollars, how many coins of each kind were in the purse?

CHAPTER VI. DERIVING AND USING FORMULAS

One thing that mathematics early imparts, unless hindered from so doing, is the idea that here, at last, is an immortality that is seemingly tangible — the immortality of a mathematical law.

— David Eugene Smith.

The Formula

Strange but true. If 100 years ago, your great-grandfather had left for you just \$1 on deposit in a bank paying 6% simple interest, you would now have exactly \$1 + \$6 interest or \$7. If, instead, he had left the dollar for you in a bank paying 6% interest compounded annually, you would now have about \$340! Do you remember the formula for simple interest? That formula was used to obtain the \$6 interest mentioned above. You probably remember from your arithmetic that the calculation of compound interest, even for only a few years, is a long and tedious process.

Imagine the labor that would be involved in finding the compound interest for 100 years! And yet the compound interest above was computed in less than one minute! You too will soon be able to perform similar feats, and even much more complicated ones, in very little time. One of the things that make this possible is the formula — in the above example, the compound interest formula. This chapter deals with this great labor-saving device, the formula.

What is a formula? A *formula* is an equation which expresses a general rule in symbols. Even the writing of a formula is accomplished in an economy of space. Two illustrations of general rules and their corresponding formulas follow.

Rule	*Formula*
The area of a rectangle is equal to the product of its length and width.	$A = lw.$
In any right triangle the square of the hypotenuse (c) is equal to the sum of the squares of the other two sides, (a) and (b).	$c^2 = a^2 + b^2.$

WRITING FORMULAS FROM VERBAL STATEMENTS

For each of the following verbal statements write a formula using the letters suggested in the problem or the initial letters of the words involved.

Group A

1. The volume of a box is the product of its length, width, and height.

2. The number of acres in a rectangular field is equal to the number of rods in its length times the number of rods in its width divided by 160.

3. The total surface (s) of a cube is 6 times the square of one edge (e).

4. The distance (d) passed over by a falling object is equal to 16 times the square of the number of seconds (t) during which it has fallen.

5. The area (A) of a rhombus is equal to one-half the product of its two diagonals (d_1 and d_2).

6. The perimeter of a rectangle is equal to the sum of twice its length and twice its width.

7. The area of a circle is equal to one-fourth the product of π and the square of its diameter.

8. The average of three numbers is one-third their sum.

9. The area (A) of a parallelogram is equal to the product of two adjacent sides (a) and (b) multiplied by the sine of the included angle (D).

10. The sine of twice angle A is equal to twice the product of the sine of A and the cosine of A.

11. The cosine of twice an angle is equal to the square of the cosine of the angle minus the square of the sine of the angle.

12. In any triangle, the square of any side (a) is equal to the sum of the squares of the other two sides (b and c) and diminished by twice the product of these two sides multiplied by the cosine of their included angle (A).

Group B

13. The surface of a closed square box is equal to the sum of twice the square of its length and 4 times the product of its length and height.

14. The selling price equals the cost plus 20% of the cost.

15. The horse-power (H. P.) of an automobile engine is equal to the number of cylinders (N) multiplied by the square of their diameter (D) divided by 2.5.

16. The volumes of two spheres (V_1) and (V_2) are to each other as the cubes of their radii (r_1 and r_2).

17. The tangent of twice an angle is equal to twice the tangent of the angle divided by the difference between 1 and the square of the tangent of the angle.

18. The sine of the sum of two angles (A) and (B) is equal to the sine of the first times the cosine of the second plus the cosine of the first times the sine of the second.

19. The cosine of the difference of two angles is equal to the product of the cosines of the angles increased by the product of the sines of the two angles.

Group C

20. The length (l) of the belt required for two pulleys, each of radius r feet, equals the circumference (c) of one pulley plus twice the distance (d) in feet between their centers.

21. The amount (A) of an investment at interest compounded annually is equal to the principal (P) multiplied by the nth power of the sum of 1 and the rate (r), n representing the number of years.

22. The sine of half angle x is equal to the square root of one-half the difference between 1 and the cosine of angle x.

23. The radius of the circumscribing circle of any triangle is equal to half the quotient of any side and the sine of the opposite angle.

WRITING VERBAL STATEMENTS FROM FORMULAS

Express each of the following formulas in words:

1. $v = 32\,t$, in which v is the velocity of a falling body and t the time in seconds.

2. $A = \frac{1}{2}\,bh$, in which A is the area of a triangle, b its base, and h its height.

3. $D = dq + r$, in which D is the dividend, d the divisor, q the quotient, and r the remainder.

4. $S = 4\,\pi r^2$, in which S is the surface of a sphere, r its radius, and π is $3\frac{1}{7}$.

5. $V = \frac{4}{3}\,\pi r^3$, in which V is the volume of a sphere and r its radius.

6. $T = \frac{1}{2}\,ab \sin C$, in which T is the area of a triangle, a and b two of its sides, and C the angle included between them.

7. $C = e^3$, in which C is the volume of a cube and e one of its edges.

8. $S = \frac{1}{2}\,h(b_1 + b_2)$, in which S is the area of a trapezoid, h its altitude, and b_1 and b_2 its bases.

9. $\sin (A - B) = \sin A \cos B - \cos A \sin B$, in which A and B are two angles.

10. $\cos (x + y) = \cos x \cos y - \sin x \sin y$, in which x and y are two angles.

From Tables to Formulas

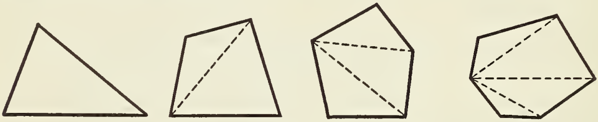
Finding formulas by experimentation. So far you have been given ready-made statements of actual practical formulas. You have probably wondered how they were discovered. The fact is, many of them were obtained as a result of actual experimentation. The results of observations made can often be summarized as a formula.

Illustrative examples.

Example 1. Find a formula for the number of triangles into which a polygon can be divided by drawing diagonals from one vertex.

Solution

Draw polygons having 3, 4, 5, and 6 sides. From one vertex of each, draw diagonals.



Into how many triangles is the 3-sided polygon divided? the 4-sided? Tabulate your results:

Number of sides (s)	3	4	5	6	7	?	17
Number of triangles (t)	1	2	3	4	?	10	?

Can you fill in the missing spaces? Notice that in every case the number of triangles (t) is 2 less than the number of sides (s). Hence, the facts may be summarized in the formula $t = s - 2$.

Something to think about. How can the formula just discovered be used to determine the values of the missing spaces in the table?

Example 2. Find the formula for the surface of a cube in terms of the length of one of its edges.

Solution

Compute the surfaces of cubes whose edges are 1 in., 2 in., 3 in., and 4 in. Tabulate your results:

Edge of cube (e)	1	2	3	4	?	10
Surface of cube (S)	6	24	54	96	150	?

The first pair of values (1 and 6) seems to indicate the formula $S = 6\,e$. Does this formula fit the second pair of values? Is this the correct formula?

Is 6 a factor of each of the surfaces? Can the table be rewritten in the form shown below?

Edge of cube (e)	1	2	3	4	5	10
Surface of cube (S)	6×1	6×4	6×9	6×16	6×25	$6\times?$

This new form shows clearly that the other factor with 6 for each surface is the square of the corresponding edge. This leads to the formula $S = 6\,e^2$. Test this formula for each pair of values and fill in the missing spaces.

EXERCISES

Group A

In each of the following examples, find the relation between the letters by inspection, complete the table of values, and express as a formula the relation existing between the letters involved.

1.

A	0	1	2	3	4	?	8	?
B	0	2	4	6	?	10	?	20

2.

x	1	2	3	4	8	?
y	3	6	9	12	?	30

3.

w	0	1	2	3	?	?	11
l	1	2	3	?	5	8	?

4.

x	0	1	2	3	4	6	?
y	0	1	4	9	16	?	64

5.

x	1	2	3	4	?	10
y	1	8	27	64	216	?

6.

m	100	81	64	49	?	1
n	10	9	8	7	5	?

7.

$\sin x$	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	?	1
$\cos y$	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	?

8.

x	0°	30°	45°	60°	?
y	90°	60°	45°	?	0°

9.

$\tan x$	1	2	3	4	5	?
$\cot x$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	?	$\frac{1}{6}$

10.

$\sin x$	0	.1	.2	.3	.4	?
$\cos y$	0	.2	.4	.6	?	1

It is not always possible to tell by inspection of a table of values the relation between the letters involved. There is a second method based on an interesting relation between the two unknowns in a linear equation.

Let us study the equation

$$3x - 2y = 6.$$

Solving for y, we obtain $y = \frac{3}{2}x - 3.$

Now let us construct our own table to fit this equation by assigning values to x and computing the corresponding values for y. Thus, if $x = 0$, $y = \frac{3}{2}(0) - 3 = -3$, etc.

x	0	2	4	6	8
y	-3	0	3	6	9

We shall now indicate the increases (positive or negative) in x and y thus:

Increase in x	$+2$	$+2$	$+2$	$+2$	
x	0	2	4	6	8
y	-3	0	3	6	9
Increase in y	$+3$	$+3$	$+3$	$+3$	

Notice that if the increase in x is always the same (and we took care that this should be so by taking $x = 0, 2, 4, 6$, etc.), then the increase in y is also always the same.

Furthermore, observe that there is a definite relationship between the increases in x and y in the table and the coefficient of x in the equation $y = \frac{3}{2}x - 3$. The coefficient of x has for its numerator the y-increase and for its denominator the x-increase, *i.e.*, the equation may be written thus:

$$y = \frac{y\text{-increase}}{x\text{-increase}}\,x - 3.$$

This will always be true for a first-degree equation in two unknowns. If we make use of these observations, we shall have another method of deriving a formula which corresponds to a table of values.

Illustrative example. Write the equation showing the relation between x and y as indicated in the following table:

x	3	5	7	9
y	17	12	7	2

Solution

Since it is difficult to tell the relation between x and y by inspection, rewrite the table indicating the increases in x and y.

Increase in x	$+2$	$+2$	$+2$	$+2$	$+2$	
x	3	5	7	9	11	?
y	17	12	7	2	?	-8
Increase in y	-5	-5	-5	-5	-5	

Since the increases in x and y are always the same, the equation must be of the first degree and of the form:

$$y = \frac{y\text{-increase}}{x\text{-increase}}\, x + C.$$

$$\therefore\ y = \frac{-5}{+2}\, x + C. \qquad (1)$$

C is yet to be determined. To find C, select any pair of values for x and y from the table; *e.g.*, $x = 3$, $y = 17$, and substitute in (1).

Thus $$17 = \frac{-5}{+2} \cdot 3 + C,$$

or $$C = \frac{49}{2}.$$

Equation (1) now becomes $y = \dfrac{-5}{+2}\, x + \dfrac{49}{2}$,

or $$2\,y = -\,5\,x + 49.$$

Something to think about. How can you prove that the equation derived is correct?

Group B

In each of the following complete the table of values and express as a formula the relation existing between the letters involved.

11.

x	1	2	3	4	8	?
y	3	5	7	9	?	23

12.

A	1	2	3	4	10	?
B	2	5	8	11	?	44

13.

d	5	7	9	11	?	27
h	0	1	2	3	6	?

14.

x	1	3	5	7	15	?
y	5	7	9	11	?	25

15.

P	0	2.5	5	7.5	10	?	25
W	0	1	2	3	4	6	?

16.

n	2	4	6	8	?	30
s	0	1	2	3	8	?

17.

y	2	4	6	8	?	20
x	10	9	8	7	4	?

18.

x	0	-2	-4	-6	-8	?	-24
y	-1	0	1	2	3	7	?

19.

tan x	0	.1	.3	.4	?	?
cot y	0	.2	.6	.8	?	2

20.

sin x	0	.1	.2	.3	.4	?	?
cos y	$-.1$	0	.1	.2	.3	?	.9

Something to think about. From the formula found in Example 20: Can cos y ever equal 1? .95? How many degrees will there be in angle y for its maximum value?

Group C

In each of the following complete the table of values and express as a formula the relation existing between the letters involved.

21.

x	0	$3a$	$6a$	$9a$	$12a$	$21a$	?
y	a	$5a$	$9a$	$13a$	$17a$	?	$41a$

22.

m	0	$5b$	$10b$	$15b$	$20b$	?	$50b$
n	$-\frac{b}{5}$	$\frac{9b}{5}$	$\frac{19b}{5}$	$\frac{29b}{5}$	$\frac{39b}{5}$	$\frac{59b}{5}$	?

23.

x	1	2	3	4	7	?
y	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	?	$\frac{1}{10}$

24.

P	-1	1	3	5	7	?	29
W	0	1	2	3	4	10	?

25.

W	1	2	3	4	?	10
C	-6	-8	-10	-12	-18	?

26.

x	0	1	2	3	4	5	?
y	0	$\frac{1}{2}$	2	$4\frac{1}{2}$	8	?	50

27.

x	0	1	2	1	7	1	10	?
y	0	1	1	2	2	10	?	12
z	0	2	3	3	9	11	15	20

28.

A	1	2	3	4	5	6	?	8	?	?
B	0	1	2	3	4	?	6	7	10	?
C	2	3	4	5	6	7	8	?	?	12

From Facts to Formulas by Generalization

Illustrative example. A regulation of an express company states that the charges for delivering packages are 10 cents for each package plus an additional charge of 1 cent for each pound of weight. Write a formula expressing the relation between the cost of delivering a package and its weight.

Solution

What is the cost of shipping a package weighing 1 lb.? 2 lb.? 3 lb.? 4 lb.? 7 lb.? 10 lb.? n lb.? Making a table, we have

Weight	1	2	3	4	7	10	n
Cost	11	12	13	14	17	20	$10 + n$

We notice that the cost is always 10 more than the weight. We can now make a formula using c for the cost and n for the number of pounds. This formula is $c = 10 + n$.

Verify this formula by checking with any set of corresponding values from the table.

Finding a relation which holds for all particular cases (*i.e.* for the nth term) is called *generalization*.

EXERCISES

Group A

1. If a teacher's salary starts at \$1500 and increases \$100 at the end of each year, what will her salary be at the end of 1 year? At the end of 2 years? 3 years? 5 years? 9 years? n years?

2. A taxicab fare is 30 cents for the first mile and 20 cents for each additional mile. What will it cost to travel 2 miles? 3 miles? 4 miles? 10 miles? n miles?

3. A telegraph company charges 25 cents for 10 words or less and 2 cents for each additional word. How much will it cost to send a telegram of 11 words? 12 words? 13 words? 14 words? 18 words? A telegram cost 55 cents. How many words did it contain? Is it possible to have a charge of 36 cents according to the above rates?

4. A girl begins a Christmas Club savings account by depositing \$6 on January 1. She agrees to deposit an additional \$4 on the first of each month through December 1st. How much will she have deposited by April 15th? June 2nd? Sept. 30th? During which month will the girl's bankbook show a deposit of \$30?

5. A printer charges \$10 for the first 1000 business cards and \$5 for each additional 1000 cards. What will be the cost for 2000 cards? 5000 cards? n-thousand cards? If the bill for printing cards is \$35, how many thousand cards were printed?

6. A boy who has \$10 in his toy bank is given permission to withdraw 10 cents each week. How much will he have left at the end of the 2nd week? 5th week? 7th week? nth week?

Group B

In each of the following exercises (*a*) make a table of values; (*b*) fill in the general term in your table; (*c*) write a formula using this general term; (*d*) verify this formula by checking with any set of corresponding values from the table.

7. A long-distance telephone call costs 50 cents for the first 2 minutes or less and 15 cents extra for each additional minute or fraction thereof.

8. The cost of sending a parcel by mail is 8 cents for the first pound and 2 cents extra for each additional pound.

9. The cost of sending a telegram is 30 cents for the first 10 words or less, and 3 cents for each additional word.

10. An automobile renting company charges $8 for the first day and $5 for each additional day.

11. For publishing a book a printer charges $500 for setting up the type and $1 extra for each book printed.

12. A circulating library charges 10 cents for the first 2 days or less, and 3 cents for each additional day.

Group C

13. A car-renting agency charges $8 for the first day, $6 for each of the next 6 days, and $4 for each additional day thereafter.

14. A gas company charges $1 for the first 200 cubic feet (or less) of gas used, and 10 cents for each 100 cubic feet thereafter.

15. An electric company charges 7 cents for each of the first 50 kilowatt hours used, 6 cents for each of the next 50 kilowatt hours, and 5 cents for each kilowatt hour thereafter.

16. A telephone company charges 5 cents for each message up to 300, $4\frac{1}{2}$ cents for each of the next 300 messages, 4 cents for each of the next 300 messages, and $3\frac{3}{4}$ cents for each call thereafter.

17. A state tax on incomes is as follows: 1% on the first $10,000; 2% on the next $40,000; 3% on amounts over $50,000.

From Pictures to Formulas

Thus far, this chapter has illustrated three ways of deriving a formula, namely: (1) from a verbal statement, (2) from a table of given values, and (3) as a generalization of particular facts. There is a fourth way, by means of the picture or graph.

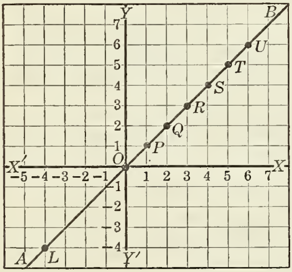
As you know, pictures in general tell stories and these stories can frequently be "read," quickly and easily. We shall now consider some mathematical pictures (*i.e.*, graphs). Each one tells its own story and that story is a formula.

Illustrative examples.

Example 1. Write the formula which corresponds to the graph of line AB shown in the graph below.

Solution

You realize by now what help is obtained from a table of values when deriving a formula. Let us form a table to fit this graph by taking the readings of a few points on line AB; *e.g.*, points P, Q, R, and S. We thus obtain the table on the right below.



Point	x	y
P	1	1
Q	2	2
R	3	3
S	4	4
T	5	?
U	?	6

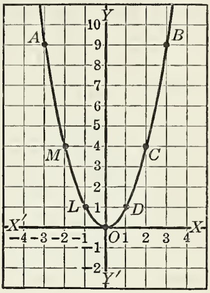
From which we see readily that the formula is $x = y$.

Check (by substitution) the accuracy of this formula by using the readings of point O; of point L.

Example 2. Write the formula which corresponds to the graph of the curve AOB as shown in the graph at the right.

Solution

Form the table by taking the readings of points B, C, D, O, L, M.



POINT	B	C	D	O	L	M
y	9	4	1	0	?	4
x	3	2	1	0	-1	?

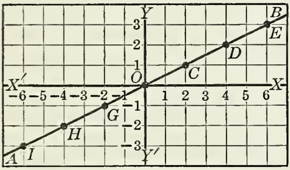
This table reveals the formula $y = x^2$.

Check this formula by substituting the readings of point A.

EXERCISES

Group A

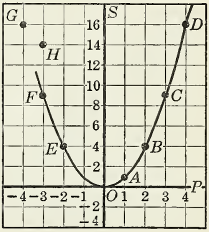
1. (*a*) From the accompanying graph, make a table of values for points C, D, E, G, H.



(*b*) Write the formula which shows the relation between x and y as indicated in the table.

(*c*) Check your formula, using the readings of point I.

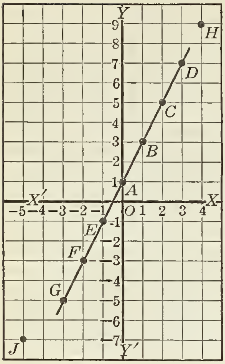
2. (*a*) From the accompanying graph, make a table of values for points O, A, B, C, D, E.



(*b*) Write the formula which shows the relation between P and S.

(*c*) Check your formula by using the readings of point F.

(*d*) Prove that the graph, if produced, will pass through point G.



(*e*) Prove that the graph, if produced, will not pass through point *H*.

3. (*a*) From the accompanying graph, make a table of values for points *A*, *B*, *C*, *D*, *E*, *F*.

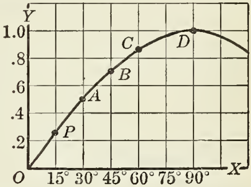
(*b*) Write the formula which shows the relation between x and y as indicated in the table.

(*c*) Check your formula by using the readings of point *G*.

(*d*) Prove that the graph, if produced, will pass through point *H*.

(*e*) Prove that the graph, if produced, will not pass through point *J*.

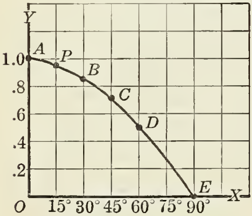
4. (*a*) From the accompanying graph, make a table of values for points *O*, *A*, *B*, *C*, *D*.



(*b*) Write the formula which shows the relation between x and y as indicated in the table. (Remember $\sin 30° = \frac{1}{2}$, $\sin 45° = \frac{1}{2}\sqrt{2}$, etc.)

(*c*) Check your formula (approximately) by using the readings of point *P*. The table of trigonometric functions shows $\sin 15° = .26$.

5. (*a*) From the accompanying graph, make a table of values for points *A*, *B*, *C*, *D*, *E*.



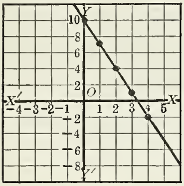
(*b*) Write the formula which shows the relation between x and y as indicated in the table. (Remember $\cos 30° = \frac{1}{2}\sqrt{3}$, $\cos 45° = \frac{1}{2}\sqrt{2}$, etc.)

(*c*) Check your formula (approximately) by using the readings of point *P*. The table of trigonometric functions shows $\cos 15° = .97$.

Group B

For each of the following graphs, write a formula which expresses the relation between x and y. Check the accuracy of each formula.

6.



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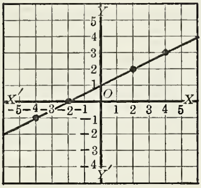

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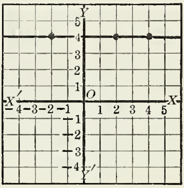

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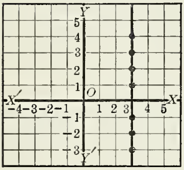

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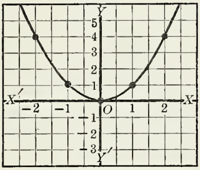

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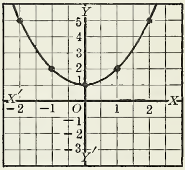
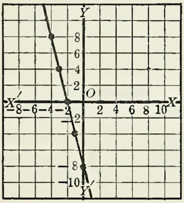


Group C

For each of the following graphs, write a formula which expresses the relation between x and y. Check the accuracy of each formula.

12.



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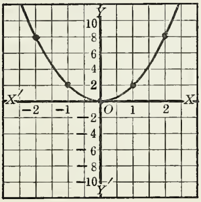

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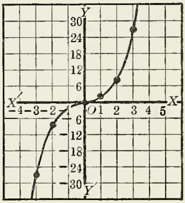


Problems

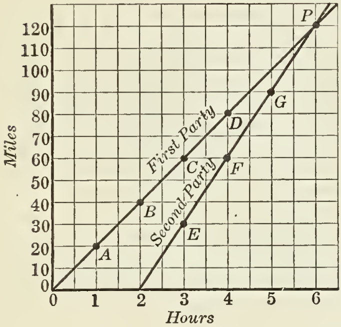
MOTION PROBLEMS SOLVED BY GRAPHS

Some types of motion problems lend themselves particularly to graphic solutions.

Illustrative examples.

Example 1. An automobile party is traveling at the rate of 20 miles per hour. A second party starts out from the same place two hours later going in the same direction, and travels at the rate of 30 miles per hour. In how many hours will the second automobile overtake the first and how far from the starting point?

Graphic Solution



On the graph, what point represents the distance the first party had gone in 1 hr.? 2 hr.? 3 hr.?

What point represents the distance the second party had gone at the end of 1 hr.? 2 hr.? 3 hr.?

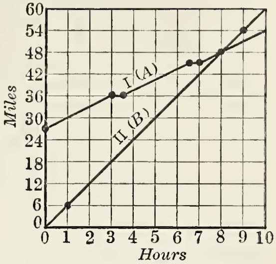
What point indicates where the second party overtakes the first? After how many hours does this take place?

How far has each party then traveled?

Answer: 1. Second party overtakes the first in 4 hours.
2. 120 miles from the starting point.

Example 2. A and B start at the same time from points 27 miles apart and travel in the same direction. If A walks at the rate of 3 miles an hour and rests $\frac{1}{2}$ hour at the end of every three hours, how fast must B travel to overtake A in 8 hours?

Graphic Solution



Broken line I pictures the progress made by A.

Since B makes no stops, begins at O and wishes to overtake A in 8 hours, his progress is pictured by the line connecting O with the 8th hour point on I. This line is II.

A study of line II shows clearly that at the end of one hour, B has covered 6 miles, at the end of 2 hours, 12 miles, etc. B's rate is therefore 6 miles an hour.

The answer can also be obtained by dividing 48 by 8. Why?

Answer: B must travel at the rate of 6 miles per hour.

EXERCISES

Solve the following problems graphically.

Group A

1. A travels 4 miles an hour. B starts from the same place 1 hour later and travels 5 miles an hour in the same direction. How many hours must B travel to overtake A? How far will they then be from the starting place?

2. A man starts from a certain place and walks at the rate of 4 miles an hour; $1\frac{1}{2}$ hours later another man starts from the same place and rides in the same direction at the rate of 8 miles an hour. In how many hours will the second man overtake the first and how far from the starting point?

3. A dirigible and an airplane are 100 miles apart when they begin to travel in the same direction. If the dirigible can make 60 miles an hour, how fast must the airplane travel in order to pass the dirigible in 5 hours?

4. A passenger train traveling 50 miles an hour leaves a certain station 3 hours after a freight train and overtakes the freight train in 7 hours. Find the rate at which the freight train is traveling.

5. A starts for a town, 120 miles away, at the rate of 20 miles an hour. His friend B, starting 2 hours later, wishes to reach the town at the same time as A. At what rate must B travel?

6. A train moving 40 miles an hour starts out 30 minutes ahead of another train moving 50 miles an hour in the same direction. How long will it take before the faster train passes the slower?

7. A man flying at the rate of 70 miles an hour starts from a flying field $2\frac{1}{2}$ hours ahead of his friend who takes off at the same field, travels in the same direction, and makes 105 miles an hour. How long will it be before the second flyer overtakes the first?

Group B

8. Two men start from New York for Albany 150 miles away, one going at the rate of 20 miles an hour and the other 30 miles an hour. If the slower man starts 2 hours earlier and rests for $\frac{1}{2}$ hour at the halfway point, who will reach Albany first?

9. An automobile leaves a town on a road which runs parallel to a railroad track at a rate of 30 miles an hour. Five hours later an express train leaves the town at a rate of 50 miles an hour and makes only one stop of 30 minutes at the end of the fourth hour of traveling. In how many hours will the train overtake the automobile? How far from the town will the train pass the automobile?

10. A travels 20 miles a day. B starts at the same time and travels 5 miles the first day, 10 miles the second day, 15 miles the third day, etc. How far from the starting point will B overtake A? How many days will it require?

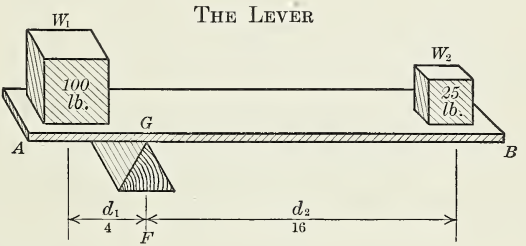
Group C

11. A takes a 30-mile hike, walking uniformly at the rate of 3 miles an hour. Show graphically that if he is to make the return journey in 5 hours, his rate must be 6 miles per hour.

12. Two boy scouts start from the same place and travel in the same direction toward their camp 30 miles away. The older boy who travels at the rate of 12 miles an hour gives the younger boy who travels 8 miles an hour a start of $1\frac{1}{2}$ hours. Who will reach the camp first?

13. Train No. 1 starts from A at 10 : 00 A.M., traveling toward a city 150 miles away at the rate of 50 miles per hour. After traveling 75 miles, the train was delayed for 30 minutes, but made no other stop. At 11 : 30 A.M. train No. 2 started out from A heading for the same city at the rate of 60 miles an hour. Which train will reach its destination first? How much sooner?

THE LEVER



In the above diagram, the *lever AB* resting on a support *FG* called a *fulcrum* is a very powerful machine. By means of a lever a great weight can be lifted with comparatively little effort. In fact, Archimedes (third century B.C.), a famous mathematician of antiquity, is said to have remarked, "Give me a fulcrum upon which to rest and I will move the earth." A rather bold statement, made, no doubt, to emphasize both the importance and simplicity of the law of the lever.

Law of the lever. In order that the lever be in balance (the weight of the lever being disregarded), the product of the weight on one side and its distance from the fulcrum must equal the product of the weight on the other side and its distance from the fulcrum.

$$w_1d_1 = w_2d_2.$$

EXERCISES

Group A

1. Henry weighing 60 lb. and his brother Fred weighing 45 lb. are playing "seesaw." If Henry sits 6 feet from the fulcrum, how far from the fulcrum must Fred sit to make the seesaw balance?

2. Jack weighs 110 lb. and sits 5 feet from the fulcrum of a teeter board (lever). What is the weight of his little brother if he sits 11 feet from the fulcrum and balances the board?

3. A, weighing 135 lb., sits $6\frac{1}{2}$ feet from the fulcrum and balances B, who sits $4\frac{1}{2}$ feet from the fulcrum. What is B's weight?

4. A boy weighing 100 lb. desires to raise an object weighing 400 lb. by means of the lever. He places one end of the lever underneath the object and the fulcrum 1 foot from the object. How far out on the lever must the boy sit in order to raise the object?

5. The weight on the long arm of a lever is 12 lb. less than the weight on the short arm. If the lengths of the arms are 6 feet and 8 feet, find the number of pounds in each weight.

Group B

6. A weight of 140 lb. on one end of a lever is balanced by 70 lb. on the other end. If one arm is 8 feet longer than the other, find the length of each arm.

7. A lever 10 yards long rests on a fulcrum which is 1.5 yards from one end of the lever. What weight will be balanced by 20 lb. placed at the end of the shorter arm?

8. A crowbar 6 feet long is used to lift a stone weighing 300 lb. Where must the fulcrum be placed in order that a weight of 100 lb. placed on the other end may just balance the stone?

9. Two boys weighing 50 lb. and 70 lb. respectively wish to balance themselves on a teeter board 12 feet long. If each boy is to sit at one end of the board, where should the fulcrum be placed?

10. In a certain rowboat, the oar is placed so that the distance from the oarlock to the water is 3 times the distance from the hand position to the oarlock. What is the propelling force of the oar at the oarlock, if one pulls on the oar with a force of 70 lb.?

Group C

11. Suppose your friend refuses to give you his weight. Explain how, knowing your own weight, you could determine with the aid of the lever your friend's weight.

12. If the weights 10 lb. and 16 lb. balance on the ends of a lever 5 feet long, find the position of the fulcrum.

13. A bar, 4 feet long, has weights of 30 lb. and 40 lb. suspended at its ends. Where must the fulcrum be placed to make it balance?

14. A false scale is one whose arms are unequal. If the arms of a false scale are $4\frac{1}{2}$ inches and 5 inches respectively and a weight of coffee on the longer arm registers 35 lb., how many pounds of coffee are actually weighed out? If the 35-pound weight were put on the longer arm, how many pounds of coffee would actually be weighed out then? Where should the fulcrum be placed to make the scale exact?

GEOMETRIC PROBLEMS

Group A

1. (*a*) If the area of a rectangle is kept constant, what happens to the width as the length is increased? decreased? doubled? halved?

(*b*) Can the length and width be increased at the same time while the area of the rectangle remains constant?

(*c*) If the length and width of a rectangle are each increased by 100%, the area of the rectangle is increased by ? %.

2. A circular pond is surrounded by a gravel path 7 feet wide. If the sum of the outer circumference of the path and the inner circumference is 132 feet, find the radius of the pond. (Use $\pi = \frac{22}{7}$.)

3. A rope 36 yards long exactly surrounds a plot of ground in the shape of a right triangle of hypotenuse 15 yards. Denote one side as x and the other as $21 - x$. Find the other two sides of the plot.

4. A rug is 4 feet longer than it is wide and has an area of 45 square feet. Find the dimensions of the rug.

5. The length of a rectangle is 4 feet longer and its width 3 feet shorter than a side of a square. The rectangle and square are equal in area. Find the dimensions of each figure.

6. The lower base of a trapezoid is 2 inches longer than the upper base; the altitude of the trapezoid is 6 inches and its area is 54 square inches. Find the lengths of the two bases.

7. The base of a triangle is twice its altitude and its area is 64. Find the base and altitude.

8. If we increase each side of a square by 3 feet, we shall increase the area by 69 square feet. Find a side of the square.

9. One leg of a right triangle is 7 feet longer than the other, and the area of the triangle is 30 square feet. Find the length of each leg.

10. The area of a circle is 50.24 square inches. Find its radius. (Use $\pi = 3.14$.)

Group B

11. An athletic field is surrounded by a circular track. A track man found that he had to run 132 feet more on the outer edge than on the inner edge to complete the circuit. Find the width of the track. ($\pi = 3\frac{1}{7}$.)

12. Find the side of a square if the number of units in its perimeter is equal to 4 times the number of units in its area.

13. A rectangle whose length is double its width would have its area increased by 100 square feet if 8 feet were added to its width and 10 feet subtracted from its length. Find the area of the rectangle.

14. A rug is 7 feet longer than it is wide and has an area of 60 square feet. Find: (*a*) the dimensions of the rug; (*b*) the distance diagonally across the rug.

15. The length of a floor exceeds its width by 3 feet. If each dimension is increased 2 feet, the area of the floor will be increased 58 square feet. Find the dimensions of the floor.

16. A rectangular piece of paper is twice as long as a square piece and 3 inches wider. The area of the rectangular piece is 216 square inches. Find a side of the square piece.

Group C

17. (*a*) The length of a rectangle is three times its width (w). Express the area of the rectangle as a function of w (*i.e.*, in terms of w).

(*b*) The altitude of a triangle is 3 inches less than the base. Express the area in terms of the base.

(*c*) The sum of the two bases of a trapezoid is 5 times the altitude h. Express the area as a function of h.

18. The width of a room is $\frac{2}{3}$ of its length. If the width had been 6 feet more, and the length 6 feet less, the room would have been square. Find the dimensions of the room.

19. The difference of the two legs of a right triangle is 7 and the hypotenuse is 17. Find the perimeter of the triangle.

20. Find the radius of a circle such that the number of feet in its circumference is equal to the number of square feet in its area.

21. The base of a parallelogram is 4 feet longer than a side and the angle between the base and this side is 30°. If the area of the parallelogram is 240 square feet, find the length of the base.

The Faithful Formula

The use of the formula as a rule for computation written in symbolic language can be traced historically back to the Ahmes papyrus, about 1650 B.C. From this document we infer that the Egyptians were concerned more with practical than theoretical results, and therefore specialized in constructing formulas for determining areas. The method used in deriving these formulas, which were inaccurate, is not known.

CUMULATIVE REVIEW

Chapters IV, V, and VI

1. Which of these statements are true? Which are false?

(*a*) $\dfrac{ax - ay}{-a} = y - x.$

(*b*) In the relation $y = \dfrac{1}{\cot A}$ if A is positive and increasing, then y is positive and decreasing.

(*c*) The roots of the equation $x^2 - 6x = 0$ are 6 and 0.

(*d*) The value of the expression $2x + \dfrac{3}{x}$ depends upon the value of x.

(*e*) In a right triangle, if $2R = \dfrac{a}{\sin A}$ and $\sin A = \dfrac{a}{c}$, then $2R = c$.

(*f*) The formula $A = bh$ is true no matter what numbers are assigned to A, b, and h.

(*g*) The formula $A = 2S$ expresses the rule: The area (A) of a square is equal to the square of one side (S).

(*h*) The relation between x and y as suggested in the table:

x	2	4	6	8	...
y	3	6	9	12	...

is expressed by the formula $\dfrac{x}{y} = \dfrac{2}{3}$.

(*i*) If a taxicab fare is 30¢ for the first mile and 20¢ for each additional mile, the cost (C) for n miles is represented by the formula $C = 30 + 20n$.

(*j*) The area of a rectangle is a function of its length and width.

2. Complete each of the following statements:

(*a*) The sum of $\dfrac{1}{\tan A - 1}$ and $\dfrac{1}{1 - \tan A}$ is ? .

(*b*) If a quantity is greater than 1, its reciprocal is ? than 1.

(*c*) $140° =$? radians.

(*d*) A can do a job in half the time B takes. If B requires 10 days to do the job, A can do $\dfrac{?}{?}$ of the job in 3 days.

(*e*) If a man invests $\frac{1}{3}$ of his capital at 5%, $\frac{2}{5}$ at 4%, and the rest at 6%, the greatest part of his capital is invested at ? %.

(*f*) If the length of a rectangle is doubled while its width is halved, the area of the rectangle is ? .

(*g*) The formula which expresses the relation between l and w, as suggested in the table:

l	1	2	3	4	...
w	3	5	7	9	...

is $w =$?

(*h*) If $\dfrac{1}{f} = \dfrac{1}{p} + \dfrac{1}{q}$, $f =$?

(*i*) The statement, "The difference between the squares of the cosine and the sine of an angle (A) is equal to the cosine of twice the angle (A)," may be expressed as a formula as follows: ? .

(*j*) A telegraph company charges x cents for the first 10 words and y cents for each additional word. The cost of sending a 12-word telegram is ? cents.

3. Change the fraction $\dfrac{\cot^2 A + 1}{\cot^2 A - 1}$ to another containing no functions other than $\sin A$.

4. Prove the identity: $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A$.

5. Solve correct to the nearest tenth:

$$\frac{5}{2x+1} - \frac{6}{1-2x} = \frac{7}{4x^2-1}.$$

6. The sides of a triangle are 8, 9, and 10. Find the number of degrees in the largest angle of the triangle.

7. A milkman has 1000 pounds of milk which tests 5% butter fat. If the law requires only 4% butter fat, how many pounds of butter fat may he remove in order to have his milk test just 4%?

8. (*a*) The cost of printing envelopes is \$4 for the first thousand, and \$2.50 for each additional thousand. What will be the cost for 2000 envelopes? 6000 envelopes?

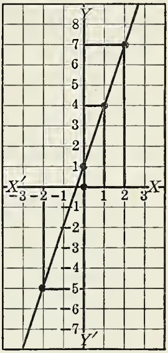
(*b*) Make a table showing what the cost is for printing envelopes from 1000 to 10,000 inclusive?

(*c*) Write a formula for the cost (c) of printing n thousand envelopes.

(*d*) Check your formula by letting $n = 5$.

(*e*) If the printing bill for envelopes is \$34, how many envelopes were printed?

9. Write the formula which expresses the relation between x and y as indicated in the graph. Check the accuracy of your formula.



Solve graphically:

10. A freighter leaves New York bound for Liverpool sailing at the rate of 15 knots an hour. Six hours later a liner leaves the same harbor sailing over the same course at the rate of 25 knots an hour. In how many hours will the liner overtake the freighter?

11. A plank 13.5 feet long is used as a seesaw by two boys who weigh 125 lb. and 100 lb. respectively. If the boys place themselves at the ends of the plank, how far from the heavier boy must the fulcrum be placed?

12. A farmer has enough wire netting to build a rectangular enclosure whose length is three times its width. He finds, however, that if he builds a square enclosure, using the same amount of netting, the area of the enclosure will be increased by 225 square feet. How many feet of netting has the farmer?

CHAPTER VII. A STUDY OF ANGLES OF ANY MAGNITUDE

It is in mathematics we ought to learn the general method, always followed by the human mind in its positive researches. — COMTE.

FUNCTIONS OF ANGLES OF ANY MAGNITUDE

Angles of any size. In Chapter I you studied the functions of angles from 0° to 90°. None of the problems, therefore, involved obtuse angles. The many practical problems based on the obtuse angle were not forgotten; they were merely reserved for this and later chapters.

Do you realize that there are angles even greater than obtuse angles? The answers to the following questions will give you an introduction to such angles.

1. Through an angle of how many degrees does the hour hand of a clock move in 1 hour? 5 hours? 18 hours? 24 hours?

2. Through an angle of how many degrees does the minute hand of a clock move in 1 minute? 25 minutes? 30 minutes? 45 minutes? 1 hour? 5 hours?

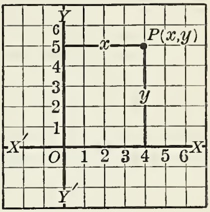
3. Through an angle of how many degrees does the spoke of a wheel move in $\frac{1}{4}$ of one revolution? $\frac{1}{3}$ of one revolution? $\frac{3}{4}$ of one revolution? 1 revolution?

4. What part of one revolution does the spoke of a wheel make in turning through an angle of 90°? 180°? 225°? 270°? 480°?

5. How many revolutions does the spoke of a wheel make in turning through an angle of 360°? 450°? 540°? 630°? 720°? 1080°?

What do you remember about graph work? Let us first review a few facts from the graph work of elementary algebra. You will remember that:

1. The two lines XX' and YY', perpendicular to each other in the same plane, are called *rectangular axes.* They are the basic lines from which measurements are made. XX' is called the *x-axis* or axis of abscissas; YY' the *y-axis* or axis of ordinates.



2. O, the point of intersection of the axes, is called the *origin.*

3. The distance of any point, such as P, from the x-axis is called the *ordinate* or *y-reading* of the point.

4. The distance of any point, such as P, from the y-axis is called the *abscissa* or *x-reading* of the point.

5. The abscissa and the ordinate of a point are the *coördinates* of the point. In the figure, point P is located by means of its coördinates (x, y) or $(4, 5)$, where 4 is the value of the x-reading and 5 the value of the y-reading. Remember that the first number inside the parenthesis is always the abscissa and the second the ordinate.

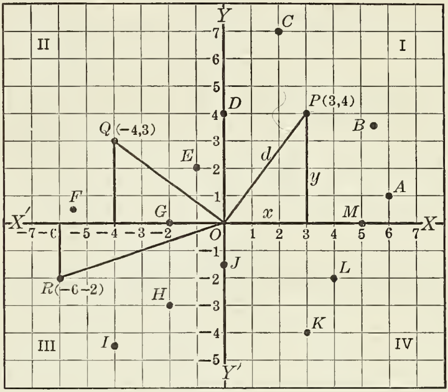
6. The axes divide the plane into four parts called *quadrants.* These quadrants are numbered I, II, III, and IV in a counterclockwise order as shown in the diagram at the top of page 160.

7. Abscissas to the right of the y-axis are positive; abscissas to the left of the y-axis are negative. Ordinates above the x-axis are positive; ordinates below the x-axis are negative.

The signs of the coördinates may be summarized:

Quadrant	I	II	III	IV
Sign of Abscissa	+	−	−	+
Sign of Ordinate	+	+	−	−

Distance of a point from the origin. If point P whose coördinates are (3, 4) is joined to the origin, the line OP represents the distance (d) of P from O. From the Pythagorean theorem, OP or $d = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$. In the same way OQ represents the distance (d) of Q from O. Here OQ or $d = \sqrt{(-4)^2 + (3)^2} = \sqrt{25} = 5$.



In general, if P is a point whose coördinates are (x, y), the distance of P from O is expressed by the positive value of d in the formula: $d = \sqrt{x^2 + y^2}$. Remember that d is always regarded as positive, no matter in what quadrant the point P is located.

EXERCISES

On graph paper, locate (plot) each of these points:

1. (7, 2). **4.** (5, − 5). **7.** (− 5, 0).

2. (− 4, 5). **5.** (9, 0). **8.** ($4\frac{1}{2}$, 1).

3. (− 2, − 3). **6.** (0, − 4). **9.** ($7\frac{1}{4}$, $10\frac{1}{2}$).

10. Read the coördinates of each of these points from the figure above: $A, B, C, D, E, F, G, H, I, J, K, L, M$.

Illustrative example. Find the distance of point R $(-6, -2)$ from the origin.

Solution

$$d = \sqrt{(-6)^2 + (-2)^2} = \sqrt{40} = \sqrt{4 \times 10} = 2\sqrt{10}.$$

Find the distance of each of the following points from the origin:

11. $(7, 0)$.
12. $(5, 12)$.
13. $(-6, -8)$.
14. $(9, -12)$.
15. $(.7, 2.4)$.
16. $(0, 0)$.
17. $(0, 5)$.
18. $(1, \frac{3}{4})$.
19. $(2, -1\frac{1}{2})$.

In each of the following examples find the length of the line joining the two given points:

20. $(5, 4)$ and $(-3, 4)$.
21. $(0, -6)$ and $(8, 0)$.
22. $(0, 0)$ and $(5, 12)$.
23. $(-3, 4)$ and $(3, -4)$.

24. (*a*) Construct the triangle whose vertices are the points $(3, 2)$, $(-3, 2)$, $(0, -2)$.

(*b*) What kind of triangle is the above?

(*c*) What is the perimeter of this triangle?

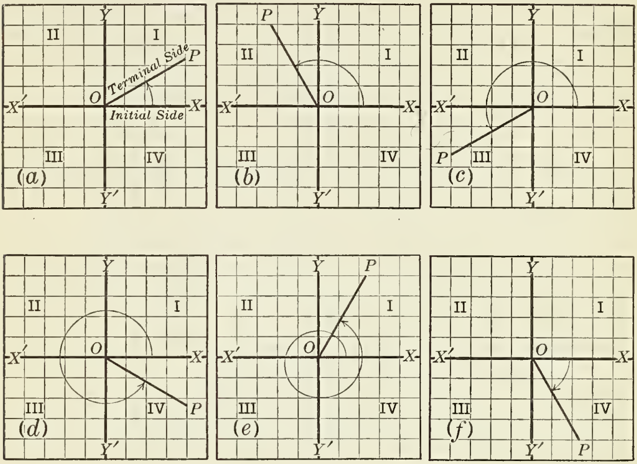
25. A square 5 units on one side is constructed in quadrant II so that one vertex of the square is at O and two of its sides coincide with the axes.

(*a*) What are the coördinates of the four vertices of the square?

(*b*) What are the coördinates of the point of intersection of the two diagonals of the square?

How angles are generated. The angle formed by two intersecting lines may be considered as the amount of rotation about the point of intersection necessary to bring one of the two lines from its position to the position of the other line. The greater the amount of rotation needed, the larger the angle. The line which is assumed to rotate is called the *terminal side* of the angle; the other line is called the *initial side*.

In each of the figures below, the angle XOP was generated by rotating the terminal side from the position OX, using the origin O as pivot, to the position of the terminal side OP. In what quadrant is the terminal side in (*a*)? (*b*)? (*c*)? (*d*)?

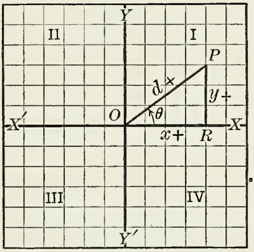


Each time the rotating line makes a complete revolution (*i.e.*, begins on OX, passes through each of the quadrants, and finally reaches OX again) an angle of 360° is generated. Since the rotating line may make as many revolutions as we please, we can obtain an angle as large as we please, *i.e.*, angles are unlimited in magnitude. For example, (*e*) shows an angle of 420°.

Positive and negative angles. If the rotating line moves counterclockwise, mathematicians have agreed to call the angle generated positive; if the rotating line moves clockwise, the angle generated is considered negative. Which of the above angles are positive? which negative?

The trigonometric functions redefined. You will recall that in Chapter I you learned the values of the trigonometric functions of an angle in terms of the opposite side, the adjacent side, and the hypotenuse of a right triangle. It is also possible to define these functions in terms of the graph work you have done in this chapter.

Let θ be a positive angle in quadrant I, OX its initial side, OP its terminal side, and $P(x, y)$ any point on the terminal side.



Then: $\sin\theta = \dfrac{y}{d} = \dfrac{\text{ordinate}}{\text{distance}}$. $\qquad \csc\theta = \dfrac{d}{y} = \dfrac{\text{distance}}{\text{ordinate}}$.

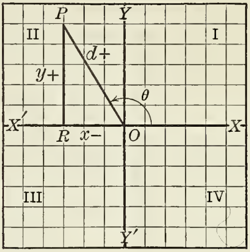
$\cos\theta = \dfrac{x}{d} = \dfrac{\text{abscissa}}{\text{distance}}$. $\qquad \sec\theta = \dfrac{d}{x} = \dfrac{\text{distance}}{\text{abscissa}}$.

$\tan\theta = \dfrac{y}{x} = \dfrac{\text{ordinate}}{\text{abscissa}}$. $\qquad \cot\theta = \dfrac{x}{y} = \dfrac{\text{abscissa}}{\text{ordinate}}$.

The signs of the trigonometric functions. As you know, abscissas and ordinates are not positive in all four quadrants. The signs of the abscissas and ordinates are used to determine the sign of each function. Thus, the signs of all functions of angles in quadrant I (acute angles) are positive. Why?

The signs of the functions of angles which terminate in other quadrants can be determined. Remember that d (the distance of any point from the origin) is always considered positive.

When θ is positive and terminates in quadrant II:

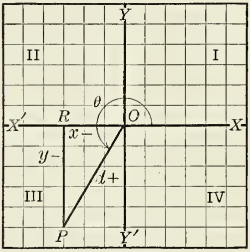


$$\sin\theta = \frac{y}{d} = \frac{+}{+} = +. \qquad \csc\theta = \frac{d}{y} = \frac{+}{+} = +.$$

$$\cos\theta = \frac{x}{d} = \frac{-}{+} = -. \qquad \sec\theta = \frac{d}{x} = \frac{+}{-} = -.$$

$$\tan\theta = \frac{y}{x} = \frac{+}{-} = -. \qquad \cot\theta = \frac{x}{y} = \frac{-}{+} = -.$$

When θ is positive and terminates in quadrant III:

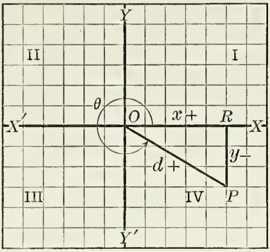


$$\sin\theta = \frac{y}{d} = \frac{-}{+} = -. \qquad \csc\theta = \frac{d}{y} = \frac{+}{-} = -.$$

$$\cos\theta = \frac{x}{d} = \frac{-}{+} = -. \qquad \sec\theta = \frac{d}{x} = \frac{+}{-} = -.$$

$$\tan\theta = \frac{y}{x} = \frac{-}{-} = +. \qquad \cot\theta = \frac{x}{y} = \frac{-}{-} = +.$$

When θ is positive and terminates in quadrant IV:

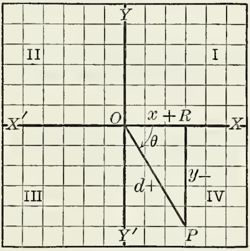


$$\sin\theta = \frac{y}{d} = \frac{-}{+} = -. \qquad \csc\theta = \frac{d}{y} = \frac{+}{-} = -.$$

$$\cos\theta = \frac{x}{d} = \frac{+}{+} = +. \qquad \sec\theta = \frac{d}{x} = \frac{+}{+} = +.$$

$$\tan\theta = \frac{y}{x} = \frac{-}{+} = -. \qquad \cot\theta = \frac{x}{y} = \frac{+}{-} = -.$$

When θ is negative and terminates in quadrant IV:



Notice that:

$$\sin\theta = \frac{y}{d} = \frac{-}{+} = -. \qquad \csc\theta = \frac{d}{y} = \frac{+}{-} = -.$$

$$\cos\theta = \frac{x}{d} = \frac{+}{+} = +. \qquad \sec\theta = \frac{d}{x} = \frac{+}{+} = +.$$

$$\tan\theta = \frac{y}{x} = \frac{-}{+} = -. \qquad \cot\theta = \frac{x}{y} = \frac{+}{-} = -.$$

Something to think about. 1. How do the signs of the functions of a negative angle in quadrant IV compare with those for the positive angle in quadrant IV?

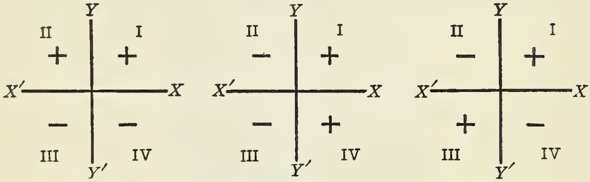
2. How do the signs of the functions of a negative angle terminating in any quadrant compare with those for the positive angle terminating in the same quadrant?

The signs of the functions of any angle may be summarized:

Summary A

QUADRANT	I	II	III	IV
Sin and csc	+	+	−	−
Cos and sec	+	−	−	+
Tan and cot	+	−	+	−

Summary B



sin and csc cos and sec tan and cot

The basic triangle. The triangle *POR*, whose sides are the abscissa, ordinate, and distance of a point *P* which lies on the terminal side of the angle, is called the *basic triangle*, since all the functions of the angle are obtained from its sides.

Constructing any angle. If we know a trigonometric function of an angle and a fact which determines the quadrant in which the angle terminates, it is always possible to construct the angle and find its other functions.

Illustrative example. Construct $\angle\theta$ if $\cos\theta = \frac{3}{5}$ and $\sin\theta$ is negative. Find the other functions of θ.

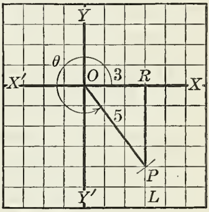
Given	*Required*
$\cos\theta = \frac{3}{5}$.	To construct $\angle\theta$.
$\sin\theta$ negative.	To find the other functions of θ.

Location of θ

Since $\cos\theta$ is positive, θ must terminate either in quadrant I or quadrant IV. Why?

Since $\sin\theta$ is negative, θ must terminate either in quadrant III or quadrant IV. Why?

$\therefore$ θ must terminate in quadrant IV.



Construction

$$\cos\theta = \frac{\text{abscissa}}{\text{distance}} = \frac{3}{5}.$$

$\therefore$ abscissa $= 3$ and distance $= 5$. In other words, in the basic triangle $OR = 3$ and $OP = 5$.

In quadrant IV construct $OR = 3$ and $RL \perp OX$. Then, with O as center and 5 as radius, cut RL at P.

Other functions of θ

$$\overline{RP}^2 = \overline{OP}^2 - \overline{OR}^2$$
$$= 25 - 9$$
$$= 16.$$
$$RP = -4.$$

$$\sin\theta = \frac{-4}{5} = -\frac{4}{5};\quad \csc\theta = -\frac{5}{4}.$$

$$\cos\theta = \frac{+3}{5} = +\frac{3}{5};\quad \sec\theta = +\frac{5}{3}.$$

$$\tan\theta = \frac{-4}{+3} = -\frac{4}{3};\quad \cot\theta = -\frac{3}{4}.$$

EXERCISES

Group A

Construct the angle, and find the remaining functions:

1. $\sin A = \frac{4}{5}$; A is an acute angle.

2. $\tan A = \frac{3}{4}$; A is an angle in quadrant I.

3. $\cos x = -\frac{6}{10}$; x is an angle in quadrant II.

4. $\tan y = -\frac{4}{3}$; y is an angle in quadrant II.

5. $\sin x = -\frac{4}{5}$; x is an angle in quadrant IV.

6. $\cos x = \frac{1}{2}$; $\tan x$ is $+$.

7. $\sin x = -\frac{3}{5}$; $\cos x$ is $-$.

8. $\tan x = 2$; $\sin x$ is $+$.

9. $\cot x = -3$; $\sin x$ is $+$.

10. $\sec y = \frac{5}{3}$; $\sin y$ is $+$.

11. $\cos A = -\frac{5}{6}$; $\cot A$ is $+$.

12. If $\sin A = \frac{1}{3}$ and A is a positive obtuse angle, find $\cos A$.

13. If $\cos A = 0.6$ and A is acute, find $\cot A$.

14. In what quadrant is $\angle A$:

(*a*) If $\sin A$ is $+$ and $\cot A$ is $-$?

(*b*) If $\cos A$ is $+$ and $\cot A$ is $-$?

(*c*) If $\sin A$ is $-$ and $\cos A$ is $+$?

(*d*) If $\sin A$ is $+$ and $\cos A$ is $-$?

(*e*) If $\sec A$ is $-$ and $\cot A$ is $+$?

15. Write the sign of each of the six functions of $-200°$; $120°$; $\frac{\pi}{2}$; $\frac{2\pi}{3}$; $\frac{11\pi}{6}$.

Group B

16. Find the other functions and construct $\angle A$, if:

(*a*) $\sin A = -\frac{\sqrt{2}}{2}$ and $\cos A$ is negative.

(*b*) $\csc A = -\sqrt{3}$ and $\cos A$ is negative.

(c) $\tan A = 2\sqrt{2}$ and $\cos A$ is negative.

(d) $\cot A = -\dfrac{1}{\sqrt{3}}$ and $\cos A$ is positive.

(e) $\cos A = -\dfrac{\sqrt{3}}{2}$ and $\tan A$ is positive.

17. $\cos x = -\frac{3}{5}$. In which quadrants may angle x be? Find the remaining functions of angle x for each of these quadrants.

18. (a) How many angles of a triangle can have a negative cosine? negative sine? negative tangent? negative cotangent? negative secant? negative cosecant?

(b) How many angles of a quadrilateral can have a negative cosine?

(c) If $\cos A$ is negative and $\tan A$ is positive, may A be an angle of a triangle?

19. Find the number of degrees in $\angle x$ (less than 360°) if x is in quadrant:

(a) II and $\sin x = \frac{1}{2}$.
(b) III and $\sin x = -\frac{1}{2}$.
(c) IV and $\cos x = \frac{1}{2}$.
(d) III and $\tan x = 1$.
(e) II and $\sin x = \dfrac{\sqrt{3}}{2}$.
(f) IV and $\sec x = 2$.
(g) III and $\csc x = -\dfrac{2}{\sqrt{3}}$.

Group C

Find the value of:

20. $2 \sin A \cos A$ if $\tan A = \frac{5}{12}$ and A is acute.

21. $\dfrac{2 \tan x}{1 - \tan^2 x}$ if $\cos x = \frac{1}{2}$ and x is in quadrant IV.

22. $\cos^2 A - \sin^2 A$ if $\sin A = -\frac{5}{12}$ and $\cos A$ is positive.

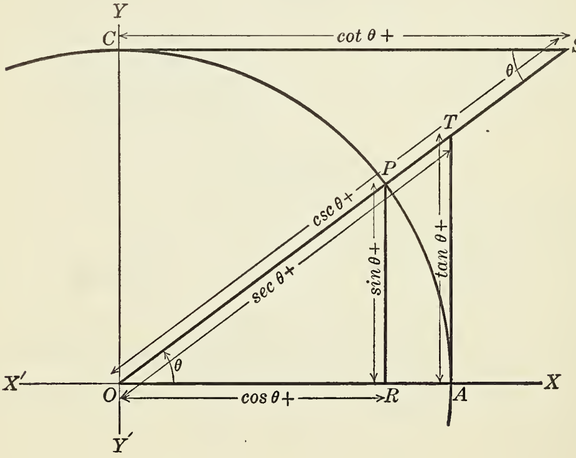
23. $\cos^2 B - \sin^2 B$ if $\cos B = \frac{1}{2}\sqrt{3}$ and $\csc B$ is negative.

24. $\dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ if $\sin x = \frac{3}{5}$ and $\cos y$ is $\frac{12}{13}$, both angles being acute.

Representing the function of any angle geometrically. As defined in Chapter I for the acute angle and again in this chapter for an angle of any size, each function of an angle is a ratio of two line segments. Thus, the six trigonometric ratios of any angle are six fractions each of which represents the indicated quotient of two lines.

Geometrically, it is possible to write each of the six fractions with a denominator equal to 1. As a result each function can be represented by one line, the numerator (since the denominator = 1). The six lines which represent geometrically the six trigonometric functions are called the *line functions* of an angle.

Line functions of an angle in quadrant I. Let θ be an angle in quadrant I. Draw a circle with O as center and with a radius equal to one unit of length. (A circle whose radius is 1 will henceforth be referred to as a *unit circle.*)



Draw $PR \perp OX$; $TA \perp OX$; $CS \perp OY$.

Then $\sin\theta = \dfrac{PR}{OP} = \dfrac{PR}{1} = PR.$

$$\cos\theta = \frac{OR}{OP} = \frac{OR}{1} = OR.$$

$$\tan\theta = \frac{PR}{OR} = \frac{TA}{OA} = \frac{TA}{1} = TA. \text{ (Rt.}\triangle POR \sim \text{Rt.}\triangle TOA.)$$

$$\cot\theta = \cot\angle OSC = \frac{CS}{OC} = \frac{CS}{1} = CS. \quad (\angle\theta = \angle OSC.)$$

$$\sec\theta = \frac{OP}{OR} = \frac{OT}{OA} = \frac{OT}{1} = OT.$$

$$\csc\theta = \csc\angle OSC = \frac{OS}{OC} = \frac{OS}{1} = OS.$$

Thus:

$\sin\theta$ = ordinate of $P = PR$.

$\cos\theta$ = abscissa of $P = OR$.

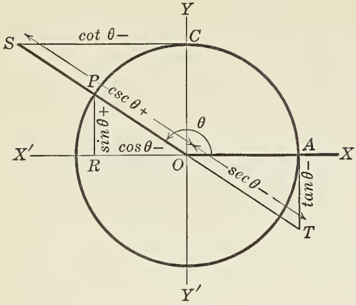
$\tan\theta$ = length of tangent from A to its intersection (T) with the terminal side, produced if necessary $= AT$.

$\cot\theta$ = length of tangent from C to its intersection (S) with the terminal side, produced if necessary $= CS$.

$\sec\theta$ = segment of the terminal side between the origin O and the intersection (T) of the terminal side, produced if necessary, and the tangent at $A = OT$.

$\csc\theta$ = segment of the terminal side between the origin O, and the intersection (S) of the terminal side, produced if necessary, and the tangent at $C = OS$.

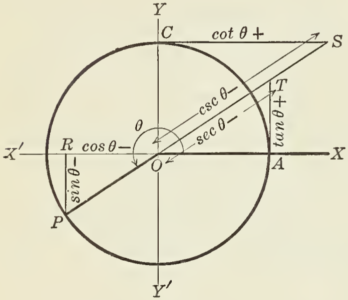
Remember that the terminal side is always considered positive; the part of the terminal side produced through O is considered negative.



Applying the summary of the line functions of any angle in quadrant I to angles in the other quadrants, we obtain:

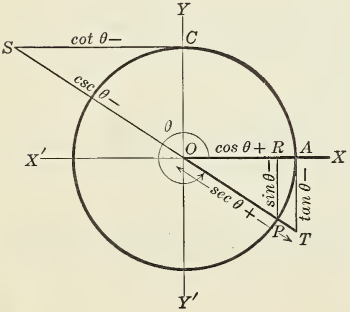
Quadrant II.

$\sin\theta = PR\ (+).$
$\cos\theta = OR\ (-).$
$\tan\theta = AT\ (-).$
$\cot\theta = CS\ (-).$
$\sec\theta = OT\ (-).$
$\csc\theta = OS\ (+).$



Quadrant III.

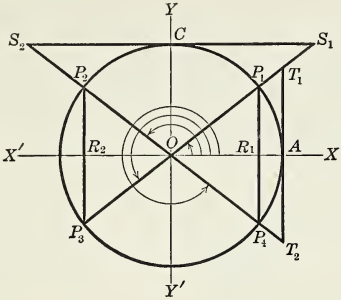
$\sin\theta = PR\ (-).$
$\cos\theta = OR\ (-).$
$\tan\theta = AT\ (+).$
$\cot\theta = CS\ (+).$
$\sec\theta = OT\ (-).$
$\csc\theta = OS\ (-).$



Quadrant IV.

$\sin\theta = PR\ (-).$
$\cos\theta = OR\ (+).$
$\tan\theta = AT\ (-).$
$\cot\theta = CS\ (-).$
$\sec\theta = OT\ (+).$
$\csc\theta = OS\ (-).$

All quadrants. The diagram below shows the line functions for an angle in any quadrant. It combines into one the four preceding diagrams.



Proofs of trigonometric relations for any angle. In Chapter IV, the following groups of relations were established for any acute angle, *i.e.*, for any angle in quadrant I.

Reciprocal Relations

$$\cot\theta = \frac{1}{\tan\theta}.$$

$$\sec\theta = \frac{1}{\cos\theta}.$$

$$\csc\theta = \frac{1}{\sin\theta}.$$

Quotient Relations

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}.$$

Pythagorean Relations

$$\sin^2\theta + \cos^2\theta = 1.$$

$$\tan^2\theta + 1 = \sec^2\theta.$$

$$\cot^2\theta + 1 = \csc^2\theta.$$

By means of the line functions just discussed, these relations can be proved true for any angle in any quadrant.

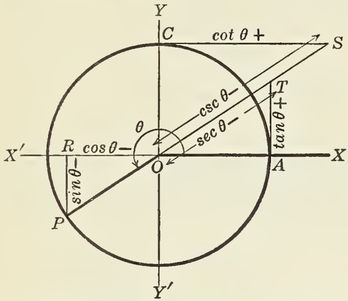
Illustration 1. Prove: $\cot\theta = \dfrac{1}{\tan\theta}$; θ being an angle in quadrant III.

Proof

Let $\angle AOP = \theta$. Draw a unit circle and construct the line functions of angle θ. Then $\cot\theta = CS$ and $\tan\theta = TA$.

Since the formula to be proved involves the tangent and cotangent functions, we shall consider the triangles that contain CS and TA as sides.

In right △ OCS and OTA, $\angle OSC = \angle AOT$.



$$\therefore \triangle OCS \sim \triangle OTA.$$

$$\therefore \frac{CS}{CO} = \frac{OA}{TA}.$$

But $CS = \cot\theta$,

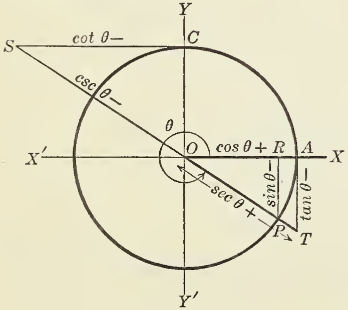
$TA = \tan\theta$,

and $CO = OA = 1$.

$$\therefore \frac{\cot\theta}{1} = \frac{1}{\tan\theta}.$$

or $$\cot\theta = \frac{1}{\tan\theta}.$$

Illustration 2. Prove: $\cot^2\theta + 1 = \csc^2\theta$, θ being an angle in quadrant IV.



Proof

Let $\angle AOP = \theta$.

In the unit circle,

$\cot\theta = CS$

and $\csc\theta = OS$.

In right $\triangle OCS$,

$$\overline{CS}^2 + \overline{CO}^2 = \overline{OS}^2.$$

But $CS = \cot\theta$,

$CO = 1$,

and $OS = \csc\theta$.

$$\therefore (\cot\theta)^2 + (1)^2 = (\csc\theta)^2.$$

or $$\cot^2\theta + 1 = \csc^2\theta.$$

Since the remaining relations can be proved by methods similar to those illustrated on page 174, it follows that all the relations (reciprocal, quotient, and Pythagorean) are true for an angle in any quadrant.

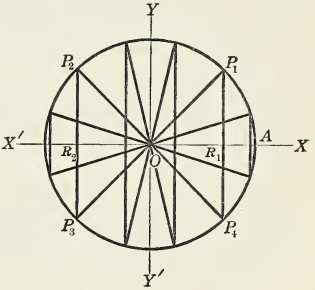
EXERCISES

By means of line functions, prove that in quadrant:

1. II, $\sec\theta = \dfrac{1}{\cos\theta}$.
2. III, $\tan^2\theta + 1 = \sec^2\theta$.
3. IV, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.
4. IV, $\csc\theta = \dfrac{1}{\sin\theta}$.
5. III, $\sin^2\theta + \cos^2\theta = 1$.
6. I, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

A Study in Changes

In Chapter I you studied the changes in the functions of an angle as the angle changed from 0° to 90°, *i.e.*, as the angle changed but remained in quadrant I. We are now ready to consider the changes in the functions of an angle as the angle changes and passes through all of the quadrants.



Changes in the sine. In the adjoining figure the circle is a unit circle. With reference to this figure complete the following statements:

1. $\sin \angle AOP_1 =$? .

2. As the angle ($\angle AOP_1$) decreases, the sine ? ; as the angle approaches 0°, the value of its sine approaches ? ; as the angle reaches 0°, its sine becomes equal to ? .

3. As the angle ($\angle AOP_1$) increases towards 90°, the sine ? ; as the angle approaches 90°, the value of its sine approaches ? ; as the angle reaches 90°, its sine becomes equal to ? .

4. Then as the angle increases from 0° to 90°, its sine is positive and *increases* from ? to ? .

5. $\sin \angle AOP_2 =$? .

6. As the angle ($\angle AOP_2$) increases from 90° to 180°, its sine ? ; as the angle approaches 180°, the value of its sine approaches ? ; as the angle reaches 180°, its sine becomes equal to ? .

7. Then as the angle increases from 90° to 180°, the sine is positive and *decreases* from ? to ? .

8. $\sin \angle AOP_3 =$? .

9. As the angle ($\angle AOP_3$) increases from 180° to 270°, its sine ? ; as the angle approaches 270°, the value of its sine approaches ? ; as the angle reaches 270°, its sine becomes equal to ? .

[When a quantity changes and assumes values between 0 and -1 inclusive, we say that the quantity *decreases* from 0 to -1. However, it should be observed that in *absolute value* (*i.e.*, the value without reference to sign) the quantity *increases* from 0 to 1.]

10. Then as the angle increases from 180° to 270°, its sine is negative and *decreases* from ? to ? .

11. $\sin \angle AOP_4 =$? .

12. As the angle ($\angle AOP_4$) increases from 270° to 360°, its sine ? ; as the angle approaches 360°, the value of its sine approaches ? ; as the angle reaches 360°, its sine becomes equal to ? .

13. Then as the angle increases from 270° to 360°, its sine is negative and *increases* from ? to ? .

Summaries for the Sine

In Quadrant	I	II	III	IV
As the angle changes from	0° to 90°	90° to 180°	180° to 270°	270° to 360°
Then the sine changes from	0 to 1	1 to 0	0 to − 1	− 1 to 0
	(increases) (positive)	(decreases) (positive)	(decreases) (negative)	(increases) (negative)

Angle	0°	90°	180°	270°	360°
Sine	0	1	0	− 1	0

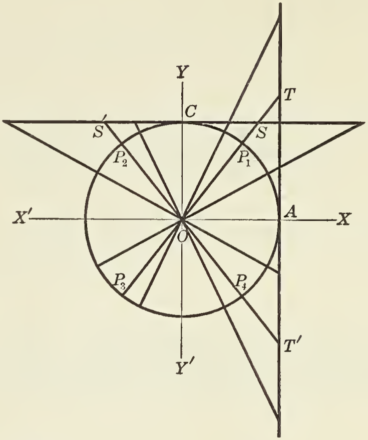
Changes in the cosine. By referring to the figure used to show the changes in the sine, show (as was done for the sine) that the following summaries are true for the cosine.

Summaries for the Cosine

In Quadrant	I	II	III	IV
As the angle changes from	0° to 90°	90° to 180°	180° to 270°	270° to 360°
Then the cosine changes from	1 to 0	0 to − 1	− 1 to 0	0 to 1
	(decreases) (positive)	(decreases) (negative)	(increases) (negative)	(increases) (positive)

Angle	0°	90°	180°	270°	360°
Cosine	1	0	− 1	0	1

Changes in the tangent. In the figure below, the circle is a unit circle. With reference to this figure, complete the following statements:



1. tan $\angle AOP_1$ = ? .

2. As the angle ($\angle AOP_1$) decreases, the tangent ? ; as the angle approaches 0°, the value of its tangent approaches ? ; as the angle reaches 0°, its tangent becomes equal to ? .

3. As the angle ($\angle AOP_1$) increases towards 90°, the tangent ? ; as the angle approaches close to 90°, its tangent ? without limit; as the angle reaches 90°, its tangent (the tangent line at A) becomes greater than any numerical value. This value is called *infinity* (∞). (Compare with page 3 in Chapter I.)

4. Then as the angle increases from 0° to 90°, its tangent is *positive* and *increases* from ? to ? .

5. tan $\angle AOP_2$ = ? .

6. As the angle ($\angle AOP_2$) decreases towards 90°, the tangent ? ; as the angle approaches close to 90°, its tangent ? with-

out limit; as the angle reaches 90°, its tangent (the tangent line at A) becomes less than any negative numerical value. This value is written as $-\infty$.

7. Thus, from 3 and 6, it follows that tan 90° = ± ? depending upon the approach. If tan 90° is approached through quadrant I, its value is ? ∞; if tan 90° is approached through quadrant II, its value is ? ∞.

(Henceforth the value of tan 90° will be written as $\pm\infty$. Furthermore, whenever any function of an angle is ∞, it may be written as $\pm\infty$.)

8. As the angle ($\angle AOP_2$) increases from 90° to 180°, its tangent in absolute value ? ; as the angle approaches 180°, the value of its tangent approaches ? ; as the angle reaches 180°, its tangent becomes equal to ? .

9. Then as the angle increases from 90° to 180°, the tangent is negative and decreases in absolute value from ? to ? .

10. tan $\angle AOP_3$ = ? .

11. As the angle ($\angle AOP_3$) increases from 180° to 270°, its tangent ? ; as the angle approaches 270°, the value of its tangent approaches + ? ; as the angle reaches 270°, its tangent becomes equal to + ? and its value is written as ± ? depending upon the approach to 270°.

12. As the angle increases from 180° to 270°, its tangent is positive and increases in absolute value from ? to ? .

13. tan $\angle AOP_4$ = ? .

14. As the angle ($\angle AOP_4$) increases from 270° to 360°, its tangent in absolute value ? ; as the angle approaches 360°, the value of its tangent approaches ? ; as the angle reaches 360°, its tangent becomes equal to ? .

15. Then as the angle increases from 270° to 360° the tangent is negative and decreases in absolute value from ? to ? .

Summaries for the Tangent

In Quadrant	I	II	III	IV
As the angle changes from	0° to 90°	90° to 180°	180° to 270°	270° to 360°
Then the tangent changes from	0 to $+\infty$	$-\infty$ to 0	0 to $+\infty$	$-\infty$ to 0
	(increases) (positive)	(decreases in absolute value) (negative)	(increases) (positive)	(decreases in absolute value) (negative)

Angle	0°	90°	180°	270°	360°
Tangent	0	$\pm\infty$	0	$\pm\infty$	0

The values of the cotangent, secant, and cosecant of 0°, 90°, 180°, 270°, and 360° can be found by the same method as that which we have used for finding the values of the sine, cosine, and tangent of these angles. However, since we know that $\cot\theta = \frac{1}{\tan\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, and $\csc\theta = \frac{1}{\sin\theta}$, we can immediately write their values for the angles 0°, 90°, 180°, 270°, 360°.

Remember

Angle	0°	90°	180°	270°	360°
sin	0	1	0	-1	0
cos	1	0	-1	0	1
tan	0	$\pm\infty$	0	$\pm\infty$	0
cot	$\pm\infty$	0	$\pm\infty$	0	$\pm\infty$
sec	1	$\pm\infty$	-1	$\pm\infty$	1
csc	$\pm\infty$	1	$\pm\infty$	-1	$\pm\infty$

EXERCISES

Group A

1. Construct the line functions of each of the following:

(*a*) $\cos 45°$. (*d*) $\tan 135°$. (*g*) $\csc 270°$.
(*b*) $\tan 60°$. (*e*) $\cos 180°$. (*h*) $\tan 300°$.
(*c*) $\sec 0°$. (*f*) $\cos 210°$. (*i*) $\cot(-30°)$.

2. Show by lines in a unit circle that in absolute value:

(*a*) $\sin 150° = \sin 30°$. (*d*) $\cot 300° = \tan 30°$.
(*b*) $\cos 120° = \cos 60°$. (*e*) $\csc 330° = \sec 60°$.
(*c*) $\tan 210° = \tan 30°$. (*f*) $\cos(-60°) = \cos 60°$.

3. Find the value of each of the following expressions:

(*a*) $x \sin 90° - y \cos 0° + x \sin 270° + y \cos 360°$.
(*b*) $2 \tan 180° - 3 \tan 0° + 2 \sec 180°$.

4. What are the largest and the smallest values possible for the: (*a*) sine? (*b*) cosine? (*c*) tangent? (*d*) cotangent? (*e*) secant? (*f*) cosecant?

5. (*a*) May $\tan 90°$ be considered as either $+\infty$ or $-\infty$? Why?

(*b*) May $\sin 90°$ be considered as either $+1$ or -1? Why?

6. (*a*) As an angle increases from 0° to 90°, which increases more rapidly, its sine or its tangent?

(*b*) As an angle increases from 0° to 90°, which decreases more rapidly, its cosine or its cotangent?

7. (*a*) If an angle is doubled, is its sine doubled?

(*b*) If an angle is doubled, is its tangent doubled?
(*c*) If an angle is halved, is its cosine doubled?

8. If θ is an angle in quadrant I, show by means of line functions which is greater:

(*a*) $\tan\theta$ or $\sin\theta$. (*c*) $\sec\theta$ or $\tan\theta$.
(*b*) $\cos\theta$ or $\cot\theta$. (*d*) $\csc\theta$ or $\cot\theta$.

Group B

9. (*a*) Can the angles of an acute triangle have negative functions?

(*b*) What functions of an angle of an obtuse triangle may be negative?

(*c*) What functions of an angle of a triangle are never negative?

10. Make a table showing the angles that have some function or functions equal to $\pm\infty$. For each angle taken, list all the functions that have a value $\pm\infty$.

11. Trace the changes in sign and magnitude of each of the following as A increases from $0°$ to $360°$:

(*a*) $1 - \sin A$. (*b*) $1 - \cos A$. (*c*) $\sin^2 A$.

Group C

12. Construct the line function of:

(*a*) $\sin 15°$. (*c*) $\sec 165°$. (*e*) $\csc 345°$.
(*b*) $\tan 75°$. (*d*) $\cos 255°$. (*f*) $\cot(-75°)$.

13. If A, B, and C are the angles of a triangle, show geometrically that:

(*a*) $\sin A = \sin(B + C)$. (*c*) $\sin \frac{1}{2} A = \cos \frac{1}{2}(B + C)$.
(*b*) $\sin(A + B + C) = 0$. (*d*) $\sin \frac{1}{2}(A + B + C) = 1$.

14. Trace the changes in sign and magnitude as A increases from $0°$ to $360°$ of each of the following:

(*a*) $\sin A \cos A$. (*c*) $\sin A + \cos A$.
(*b*) $\sin A - \cos A$. (*d*) $\tan A - \cot A$.

Reduction Formulas

You will probably be surprised to learn that the table of values of the functions of acute angles (page 3) is the very one that can be used to find the functions of an angle of any size terminating in any of the quadrants. In order to use this table for an angle of any size, we shall have to derive a few reduction formulas, to show how to reduce the function of any angle to the function of an acute angle.

First complete the following examples:

1. $91° = 180° - ?$. $\qquad 136°\,15' = 180° - ?$.
$100° = 180° - ?$. $\qquad 141°\,49' = 180° - ?$.
$125° = 180° - ?$. $\qquad 176°\,13' = 180° - ?$.

Thus an angle in quadrant II can be expressed as the difference between 180° and an ? angle in quadrant I.

2. $215° = 180° + ?$. $\qquad 203°\,15' = 180° + ?$.
$269° = 180° + ?$. $\qquad 220°\;\;1' = 180° + ?$.
$240° = 180° + ?$. $\qquad 245°\,10' = 180° + ?$.

Thus an angle in quadrant III can be expressed as the sum of 180° and an ? angle in quadrant I.

3. $295° = 360° - ?$. $\qquad 320°\,15' = 360° - ?$.
$310° = 360° - ?$. $\qquad 345°\,10' = 360° - ?$.
$345° = 360° - ?$. $\qquad 350°\,14' = 360° - ?$.

Thus an angle in quadrant IV can be expressed as the difference between 360° and an ? angle in quadrant I.

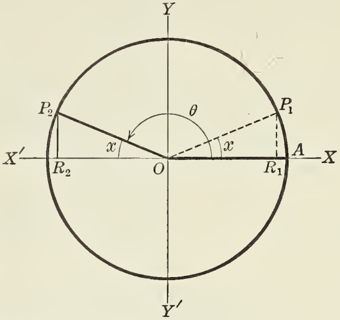
4. For each of the following negative angles, write the number of degrees in the positive angle formed by the same two sides that make the negative angle:

$-20°$	$-110°$	$-250°$	$-275°$
$-90°$	$-345°$	$-85°$	$-115°$
$-315°$	$-200°$	$-60°\,35'$	$-116°\,42'$

Thus a negative angle of x degrees is determined by the same initial and terminal sides as the positive angle of ? degrees.

Reducing functions of an angle in quadrant II. It is possible to reduce the functions of $(180° - x)$, an angle in quadrant II, to functions of x, an angle in quadrant I.

In the unit circle O, let $\angle AOP_2 = \theta = 180° - x$ and $\angle AOP_1 = \angle X'OP_2 = x$.



Draw P_1R_1 and $P_2R_2 \perp XX'$.

$$\text{Then } \triangle OP_2R_2 \cong \triangle OP_1R_1.$$

$$P_2R_2 = P_1R_1, \text{ and } OR_2 = -\ OR_1.$$

$$\sin(180° - x) = P_2R_2 = P_1R_1 = +\sin x.$$

$$\cos(180° - x) = OR_2 = -\ OR_1 = -\cos x.$$

$$\tan(180° - x) = \frac{P_2R_2}{OR_2} = \frac{P_1R_1}{-\ OR_1} = -\tan x.$$

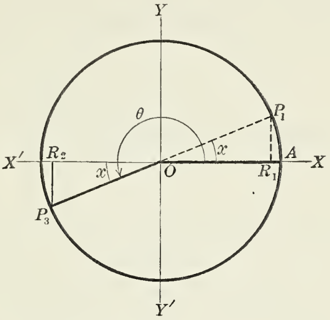
$$\cot(180° - x) = \ ? \ = \ ? \ = -\cot x.$$

$$\sec(180° - x) = \ ? \ = \ ? \ = -\sec x.$$

$$\csc(180° - x) = \ ? \ = \ ? \ = +\csc x.$$

Reducing functions of an angle in quadrant III. It is also possible to reduce the functions of $(180° + x)$, an angle in Q III (Q means quadrant), to functions of x, an angle in Q I.

In the unit circle O, let $\angle AOP_3 = \theta = 180° + x$ and $\angle AOP_1 = \angle X'OP_3 = x$.



Draw P_1R_1 and $P_3R_2 \perp XX'$.

Then: $\triangle OR_2P_3 \cong \triangle OP_1R_1$

$$P_3R_2 = - P_1R_1, \text{ and } OR_2 = - OR_1.$$

$$\sin(180° + x) = P_3R_2 = - P_1R_1 = - \sin x.$$

$$\cos(180° + x) = OR_2 = - OR_1 = - \cos x.$$

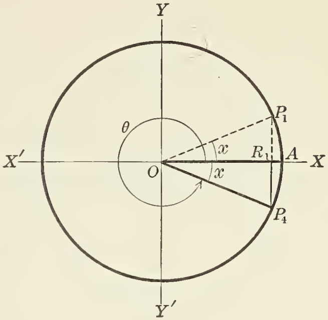
$$\tan(180° + x) = \frac{P_3R_2}{OR_2} = \frac{- P_1R_1}{- OR_1} = + \tan x.$$

$$\cot(180° + x) = \ ? \ = \ ? \ = + \cot x.$$

$$\sec(180° + x) = \ ? \ = \ ? \ = - \sec x.$$

$$\csc(180° + x) = \ ? \ = \ ? \ = - \csc x.$$

Reducing functions of an angle in quadrant IV. It is also possible to reduce the functions of $(360° - x)$, an angle in Q IV to functions of x, an angle in Q I. Show, as in the previous illustrations, that $\triangle OR_1P_4 \cong \triangle OP_1R_1$ and that $P_4R_1 = -P_1R_1$.



Then:

$$\sin(360° - x) = P_4R_1 = -P_1R_1 = -\sin x.$$
$$\cos(360° - x) = OR_1 = +\cos x.$$
$$\tan(360° - x) = ? = ? = -\tan x.$$
$$\cot(360° - x) = ? = ? = -\cot x.$$
$$\sec(360° - x) = ? = ? = +\sec x.$$
$$\csc(360° - x) = ? = ? = -\csc x.$$

The functions of a negative angle can be reduced to functions of a positive angle. Since a rotation through 360° leaves an angle exactly in its initial position, the functions of $-x$ are exactly the same as those of $360° - x$ given above and can be reduced to functions of the positive angle x in exactly the same way.

Similarly, in dealing with angles larger than 360°, we can subtract 360° or any multiple of 360° without changing any of the functions of the angle.

Summary of Reduction Formulas

Any function of an angle expressed as $\begin{bmatrix} 180° - x \\ 180° + x \\ 360° - x \end{bmatrix}$ *(x being acute) equals the same-named function of x, prefixed by the sign of that function in the quadrant in which the original angle terminates.*

Illustrative examples.

Example 1. Express sin 116° as a function of a positive acute angle.

Solution

Since 116° terminates in quadrant II, sin 116° is +.

$$\sin 116° = \sin (180° - 64°) = + \sin 64°.$$

Example 2. Express sin 310° as a function of a positive acute angle.

Solution

Since 310° terminates in quadrant IV, sin 310° is −.

$$\sin 310° = \sin (360° - 50°) = - \sin 50°.$$

Example 3. Express tan 239° as a function of a positive angle less than 45°.

Solution

Since 239° terminates in quadrant III, tan 239° is +.

$$\tan 239° = \tan (180° + 59°) = + \tan 59°$$
$$= + \cot 31°.$$ (See page 18.)

Example 4. Find the value of csc 210°.

Solution

Since 210° terminates in quadrant III, csc 210° is −.

$$\csc 210° = \csc (180° + 30°) = - \csc 30° = - 2.$$

Example 5. Find the value of $\cos 420°$.

Solution

$$\cos 420° = \cos(360° + 60°) = \cos 60° = \tfrac{1}{2}.$$

Example 6. Find the value of $\cot \frac{5\pi}{3}$.

Solution

$$\cot \frac{5\pi}{3} = \cot \frac{5 \times 180°}{3} = \cot 300°$$
$$= \cot(360° - 60°) = -\cot 60° = -\tfrac{1}{3}\sqrt{3}.$$

Example 7. Express $\sec(-41°)$ as a function of a positive acute angle.

Solution

$$\sec(-41°) = \sec(360° - 41°) = +\sec 41°.$$

EXERCISES

1. Express each of the following as a function of a positive acute angle:

(*a*) $\sin 105°$.
(*b*) $\cos 170°$.
(*c*) $\sec 340°$.
(*d*) $\cot 242°$.
(*e*) $\csc 290°$.
(*f*) $\tan 184°$.
(*g*) $\cos 190°$.
(*h*) $\tan 118°$.
(*i*) $\sin(-200°)$.
(*j*) $\cos 350°$.
(*k*) $\sec(-112°)$.
(*l*) $\cot(-115°)$.
(*m*) $\tan 615°$.
(*n*) $\cot 712°$.
(*o*) $\tan 97° 30'$.
(*p*) $\sin 241° 7'$.
(*q*) $\cos 268° 9'$.
(*r*) $\csc 173° 40'$.
(*s*) $\cot(-11°)$.
(*t*) $\sec 152° 30'$.
(*u*) $\sec \frac{2\pi}{3}$.
(*v*) $\cos \frac{5\pi}{4}$.
(*w*) $\csc \frac{3\pi}{2}$.
(*x*) $\sin \frac{5\pi}{3}$.
(*y*) $\tan \frac{3\pi}{5}$.

2. Express each of the following as a function of a positive angle less than 45°:

(*a*) $\sin 105°$.
(*b*) $\cos 192°$.
(*c*) $\tan 285°$.
(*d*) $\cot(-260°)$.
(*e*) $\sec 650°$.
(*f*) $\csc 600°$.
(*g*) $\cos 120°$.
(*h*) $\tan 261°$.
(*i*) $\cot 309°$.

(*j*) sec 408°.
(*k*) csc (− 49°).
(*l*) sin (− 302°).
(*m*) tan 134°.
(*n*) cot 254°.
(*o*) sec (− 46°).
(*p*) csc (− 240°).
(*q*) sin 112° 5′.
(*r*) cos 81° 57′.
(*s*) cot 99° 18′.
(*t*) sec 147° 2′.
(*u*) sin 1.4 π.
(*v*) csc $\frac{7\pi}{10}$.
(*w*) cos $\frac{13\pi}{8}$.
(*x*) tan $\frac{6\pi}{5}$.

3. Find the value (in radical form) of each of the following:

(*a*) cot 120°.
(*b*) cos 210°.
(*c*) sin 240°
(*d*) sec (− 45°).
(*e*) sin 135°.
(*f*) tan 300°.
(*g*) cot (− 120°).
(*h*) csc 225°.
(*i*) sec 330°.
(*j*) csc (− 315°).
(*k*) cos 690°.
(*l*) cot 330°.
(*m*) tan 240°.
(*n*) sin 600°.
(*o*) cot $\frac{7\pi}{6}$.
(*p*) cos $\frac{5\pi}{6}$.

4. Using the table on page 3, find the value of each of the following:

(*a*) cos 93°.
(*b*) sin 175°.
(*c*) cot (− 33°).
(*d*) tan 107°.
(*e*) sin 309°.
(*f*) cot 111°.
(*g*) sin 98° 48′.
(*h*) cos 162° 1′.
(*i*) tan 201° 3′.
(*j*) cot 201° 3′.
(*k*) tan 94° 6′.
(*l*) cos $\frac{9\pi}{5}$.

5. Find all positive values of x less than 360° if:

(*a*) $\sin x = +\frac{1}{2}$.
(*b*) $\cot x = -1$.
(*c*) $\cos x = -\frac{1}{2}$.
(*d*) $\tan x = 2$.
(*e*) $\sin x = -1$.
(*f*) $\cot x = \frac{2}{3}$.
(*g*) $\cos x = \frac{1}{2}\sqrt{2}$.
(*h*) $\tan x = .78$.
(*i*) $\sin x = \pm\frac{3}{4}$.

Another way of representing angles in the various quadrants. You have already seen that if x is an acute angle,

an angle in quadrant II can be represented as $180° - x$;
an angle in quadrant III can be represented as $180° + x$;
an angle in quadrant IV can be represented as $360° - x$.

The reduction formulas for these angles have just been considered.

Similarly, you can readily understand that if y is an acute angle,

an angle in quadrant II can be represented as $90° + y$;
an angle in quadrant III can be represented as $270° - y$;
an angle in quadrant IV can be represented as $270° + y$.

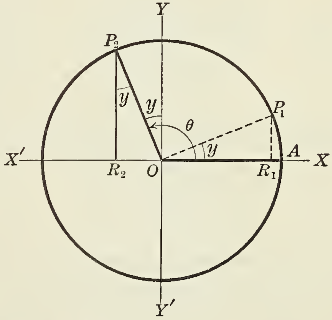
EXERCISES

Complete the following:

1. If 100° is to be written in the form $90° + y$, $y = ?$
2. If 195° is to be written in the form $270° - y$, $y = ?$
3. If 260° is to be written in the form $270° - y$, $y = ?$
4. If 300° is to be written in the form $270° + y$, $y = ?$

Reducing functions of $90° + y$. Below is a development of the method used to reduce the functions of $(90° + y)$, an angle in Q II, to functions of y, an angle in Q I.

In the unit circle O, let $\angle AOP_2 = \theta = 90° + y$. Then $\angle P_2OY = y$.



Make $\angle AOP_1 = y = \angle OP_2R_2$.

Since $P_2R_2 \perp XX'$, $\angle OP_2R_2 = \angle P_2OY = \angle P_1OR_1 = y$ and $\triangle OP_2R_2 \cong \triangle OP_1R_1$.

$\therefore P_2R_2 = OR_1$ and $OR_2 = - P_1R_1$.

Then:

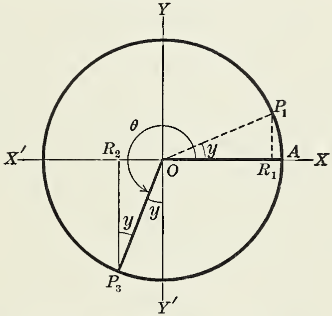
$$\sin(90^\circ + y) = P_2R_2 = OR_1 = +\cos y.$$
$$\cos(90^\circ + y) = OR_2 = -P_1R_1 = -\sin y.$$
$$\tan(90^\circ + y) = \frac{P_2R_2}{OR_2} = \frac{OR_1}{-P_1R_1} = -\cot y.$$
$$\cot(90^\circ + y) = ? = ? = -\tan y.$$
$$\sec(90^\circ + y) = ? = ? = -\csc y.$$
$$\csc(90^\circ + y) = ? = ? = +\sec y.$$

Reducing functions of $270^\circ - y$. Below is a development of the method used to reduce the functions of $(270^\circ - y)$, an angle in Q III, to functions of y, an angle in Q I.

In the unit circle O, let $\angle AOP_3 = \theta = 270^\circ - y$. Then $\angle P_3OY' = y = \angle OP_3R_2$. Make $\angle AOP_1 = y$.

$\triangle OR_2P_3 \cong \triangle OP_1R_1$.

$\therefore P_3R_2 = -OR_1$ and $OR_2 = -P_1R_1$.



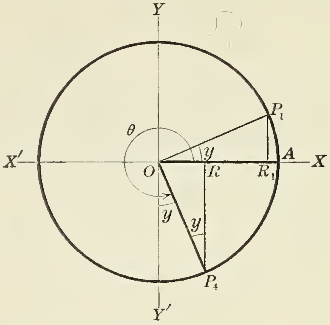
Then:

$$\sin(270^\circ - y) = P_3R_2 = -OR_1 = -\cos y.$$
$$\cos(270^\circ - y) = OR_2 = -P_1R_1 = -\sin y.$$
$$\tan(270^\circ - y) = ? = ? = +\cot y.$$
$$\cot(270^\circ - y) = ? = ? = +\tan y.$$
$$\sec(270^\circ - y) = ? = ? = -\csc y.$$
$$\csc(270^\circ - y) = ? = ? = -\sec y.$$

Reducing functions of $270° + y$. Below is a development of the method used to reduce the functions of $(270° + y)$, an angle in Q IV, to functions of y, an angle in Q I.

$\triangle ORP_4 \cong \triangle OP_1R_1$. Why?

$\therefore\ OR = P_1R_1$, and $P_4R = -\ OR_1$.



Then:

$$\sin\,(270° + y) = P_4R = -\ OR_1 = -\cos y.$$
$$\cos\,(270° + y) = OR = +\ P_1R_1 = +\sin y.$$
$$\tan\,(270° + y) = \ ?\ = \ ?\ = -\cot y.$$
$$\cot\,(270° + y) = \ ?\ = \ ?\ = -\tan y.$$
$$\sec\,(270° + y) = \ ?\ = \ ?\ = +\csc y.$$
$$\csc\,(270° + y) = \ ?\ = \ ?\ = -\sec y.$$

Summary of Reduction Formulas

Any function of an angle expressed as $\begin{bmatrix} 90° + y \\ 270° - y \\ 270° + y \end{bmatrix}$ *(y being acute) equals the co-named function of y, prefixed by the sign of that function in the quadrant in which the original angle terminates.*

Illustrative examples.

Example 1. Express sin 126° as a function of a positive acute angle.

Solution

Since 126° terminates in quadrant II, sin 126° is +.

$$\sin 126° = \sin(90° + 36°) = +\cos 36°.$$

Example 2. Express cos 221° as a function of a positive angle less than 45°.

Solution

Since 221° terminates in quadrant III, cos 221° is −.

$$\cos 221° = \cos(270° - 49°) = -\sin 49° = -\cos 41°.$$

Example 3. Find the value of tan 240°.

Solution

$$\tan 240° = \tan(270° - 30°) = \cot 30° = \sqrt{3}.$$

Example 4. Find the value of $\sec \frac{11\pi}{6}$.

Solution

$$\sec \frac{11\pi}{6} = \sec \frac{11 \times 180°}{6} = \sec 330°.$$

$$\sec 330° = \sec(270° + 60°) = \csc 60° = \tfrac{2}{3}\sqrt{3}.$$

EXERCISES

1. Express each of the following as a function of an acute angle:

(*a*) sin 117°.	(*f*) cot 183°.	(*k*) cos 98° 47′.
(*b*) tan 205°.	(*g*) csc 169°.	(*l*) sec 209.5°.
(*c*) cos 197°.	(*h*) sin 355°.	(*m*) tan 260° 40′.
(*d*) $\sec \frac{3\pi}{5}$.	(*i*) $\sin \frac{11\pi}{10}$.	(*n*) $\cot \frac{7\pi}{4}$.
(*e*) $\tan \frac{9\pi}{10}$.	(*j*) $\cos \frac{3\pi}{4}$.	(*o*) $\csc \frac{3\pi}{5}$.

2. Express each of the following as a function of a positive angle less than 45°:

(*a*) sec 115°. (*e*) sin 280°. (*i*) sin 99° 59′.
(*b*) cot 261°. (*f*) tan 211°. (*j*) cos 283° 7′.
(*c*) csc 290°. (*g*) cos 168°. (*k*) tan 152° 6′.
(*d*) tan 1.6π. (*h*) cos $\frac{7\pi}{6}$. (*l*) cot $\frac{4\pi}{3}$.

3. Without tables, find the value of each of the following:

(*a*) sin 300°. (*d*) csc 225°. (*g*) cot 330°. (*j*) sec 120°.
(*b*) cos 120°. (*e*) tan 210°. (*h*) sin 135°. (*k*) csc 240°.
(*c*) tan 315°. (*f*) sin $\frac{5\pi}{4}$. (*i*) cos $\frac{5\pi}{3}$. (*l*) cot 1.75π.

EXERCISES

Group A

1. Which of the six functions has the same value for both of the angles:

(*a*) 30°, 150°? (*c*) 390°, − 210°? (*e*) 120°, − 120°?
(*b*) 45°, − 135°? (*d*) 240°, 480°? (*f*) 225°, 405°?

2. If x is an acute angle, which of these statements are true? Which are false?

(*a*) $\cos(180° + x) = \cos(180° - x)$.
(*b*) $\tan(90° + x) = \cot(180° - x)$.
(*c*) $\sin(x - 180°) = \sin x$. (*e*) $\tan(x - 180°) = -\tan x$.
(*d*) $\cos(x - 180°) = \cos x$. (*f*) $\cot(x - 180°) = -\cot x$.

Group B

3. Simplify each of the following expressions:

(*a*) $\sin(90° + B) - \sin(90° - B)$
$- \sin(180° - B)\sin(180° + B)$.

(*b*) $\cos^2(180° - x) - \sin^2(180° - x)$.

(c) $\cos(180° + A)\cos(90° + A)$
$+ \sin(180° + A)\sin(90° + A)$.

4. Is there any one of the six functions which has the same value for each of the angles:

(a) $120°, -120°, -240°, 480°$?
(b) $36°, 144°, 396°, -216°$?

Group C

5. If A, B, and C are the three angles of a triangle (A being acute), prove that:

(a) $\sin A = \sin(B + C)$.
(b) $\cos A = -\cos(B + C)$
(c) $\tan A = -\tan(B + C)$.
(d) $\sin A = -\sin(2A + B + C)$.

6. If x is an acute angle, simplify each of the following:

(a) $\sin(540° + x)$.
(b) $\cos(450° + x)$.
(c) $\tan(360° + x)$.
(d) $\sin(x - 90°)$.
(e) $\tan(x - 180°)$.
(f) $\cos(-x - 180°)$.

Application of Reduction Formulas to the Solution of Trigonometric Equations

You are now ready to solve completely trigonometric equations. Up to this point you have been able to determine only one answer (the angle in quadrant I) to each equation given. But now you will learn that trigonometric equations, in general, have more than one root or answer. The reduction formulas enable you to obtain all the answers to any trigonometric equation.

While no general method can be given for the solution of trigonometric equations, the following general suggestions will prove helpful.

1. Reduce the expressions to the same function; or, if this is not easily accomplished, change all the functions into sines and cosines.

2. Clear of fractions.

3. Now change all the functions to the same function (if not already done in step 1).

4. Solve the equation algebraically, considering the one function occurring in the equation as the unknown quantity.

5. Write the angles which satisfy the values of the function as found in step 4. Here, you must remember the summary below, which follows from the reduction formulas.

First find a value of x in Q I as if the function were positive.

Then if	One answer is in quadrant	expressed as	The other answer is in quadrant	expressed as
$\sin x$ is +	I	x	II	$180° - x$
$\sin x$ is −	III	$180° + x$	IV	$360° - x$
$\cos x$ is +	I	x	IV	$360° - x$
$\cos x$ is −	II	$180° - x$	III	$180° + x$
$\tan x$ is +	I	x	III	$180° + x$
$\tan x$ is −	II	$180° - x$	IV	$360° - x$

Note: 1. Unless otherwise stated, only answers less than 360° should be found.

2. Since the values of the functions of angles differing by 360° or multiples thereof are exactly the same, any number of answers may be written beyond the limits of the above table, by adding 360°, 720°, 1080°, etc., to the answers below 360° already obtained. Thus if 30° and 150° are answers to an equation, other sets of answers are $(360° + 30°)$ and $(360° + 150°)$; $(720° + 30°)$ and $(720° + 150°)$; etc.

3. The cosecant, secant, and cotangent functions are not included in the above table because they can be converted to the sine, cosine, and tangent functions by means of the reciprocal relations.

6. Check all answers in the original equation.

Illustrative examples.

Example 1. Solve $\tan^2 x - \sec x = 1.$

Solution

$$\begin{aligned} \tan^2 x - \sec x &= 1. \\ \sec^2 x - 1 - \sec x &= 1. \qquad [\tan^2 x = \sec^2 x - 1.] \\ \sec^2 x - \sec x - 2 &= 0. \\ (\sec x - 2)(\sec x + 1) &= 0. \end{aligned}$$

$$\begin{aligned} \sec x &= 2. \\ \therefore \cos x &= \tfrac{1}{2}. \\ x_1 &= 60°. \\ x_2 = 360° - 60° &= 300°. \end{aligned}$$

[Since $\cos x$ is $+$, one value is in Q I, (x), and another is in Q IV, $(360° - x)$.]

$$\begin{aligned} \sec x &= -1. \\ \therefore \cos x &= -1. \\ x_3 &= 180°. \end{aligned}$$

[Why are there not two answers from this value of $\cos x$?]

$$x = 60°, 300°, 180°$$

Check

$$\begin{aligned} x &= 60°. \\ (\tan 60°)^2 - \sec 60° &= 1. \\ (\sqrt{3})^2 - 2 &= 1. \\ 1 &= 1. \end{aligned}$$

$$\begin{aligned} x &= 300°. \\ (\tan 300°)^2 - \sec 300° &= 1. \\ (-\sqrt{3})^2 - 2 &= 1. \\ 1 &= 1. \end{aligned}$$

$$\begin{aligned} x &= 180° \\ (\tan 180°)^2 - \sec 180° &= 1. \\ 0 - (-1) &= 1. \\ 1 &= 1. \end{aligned}$$

Something to think about.

1. If an equation reduces to the form $\sin x = \pm a$, where a is a number greater than 1, what can you say about values of x?

2. Of what other function is this true?

3. Show how this applies to the solution of the equation:

$$2 \cos^2 x + 7 \cos x = 0.$$

Example 2. Solve $6 \cot x = \tan x - 5$.

Solution

$$6 \cot x = \tan x - 5.$$

$$\frac{6}{\tan x} = \tan x - 5.$$

$$6 = \tan^2 x - 5 \tan x.$$

$$\tan^2 x - 5 \tan x - 6 = 0.$$

$$(\tan x + 1)(\tan x - 6) = 0.$$

$\tan x = -1.$

$\therefore x_1 = 180° - 45° = 135°.$

$x_2 = 360° - 45° = 315°.$

[Since $\tan x$ is $-$, one value is in Q II, $(180° - x)$, and another is in Q IV, $(360° - x)$.]

$\tan x = 6.$

$\therefore x_3 = 81°.$

$x_4 = 180° + 81° = 261°.$

[Since $\tan x$ is $+$, one value is in Q I, (x), and another is in Q III, $(180° + x)$, correct to the nearest degree.]

$$x = 135°, 315°, 81°, 261°$$

Check in the original equation. When verifying, the first two answers will check, and the last two, being correct only to the nearest degree, will check approximately.

EXERCISES

Group A

Solve the following equations for all positive values of the unknown angle less than 360°, and check:

1. $2 \sin x - 1 = 0.$
2. $\tan x = -1.$
3. $\cos^2 x = 1.$
4. $\tan^2 x = 1.$
5. $4 \cos^2 x = 1.$
6. $2 \sin^2 x = 1.$
7. $\sec^2 x = 2.$
8. $\tan^2 x = 3.$
9. $\sin x \cos x = 0.$
10. $\sin y(\sin y - 1) = 0.$
11. $\tan^2 y - \tan y = 0.$
12. $2 \sin^2 x + 5 \sin x = 3.$
13. $\tan A - 3 \cot A = 0.$
14. $3 \cos^2 x = \sin^2 x.$
15. $\tan A \csc A = 1.$
16. $\sin \theta + 2 \sin \theta \cos \theta = 0.$

17. $5 \tan^2 x - \sec^2 x = 11$.

18. $\cos \theta + \cos^2 \theta - \sin^2 \theta = 0$.

19. $\tan^2 A - 2 + \cot^2 A = 0$.

20. $2 \sin^2 y = 3 \cos y$.

21. $\sin A = \cos^2 A - \sin^2 A$.

22. $\tan x + \cot x = 2$.

23. $1 + \cos A = 2 \cos^2 A$.

24. $2 \cos x - 5 + 2 \sec x = 0$.

25. $2 \sin \theta - \tan \theta = 0$.

26. $\cos \theta + \tan \theta = \sec \theta$.

27. $2 \sin^2 B - \sin B = 0$.

28. $2 \cos^2 B - \cos B = 0$.

29. $2 \sin^2 x - 3 \sin x + 1 = 0$.

30. $\csc^2 \theta - \csc \theta = 0$.

31. $\sec^2 \theta - 2 \sec \theta = 0$.

32. $2 \sin \theta + \csc \theta = 3$.

33. $\left(\sin x + \frac{1}{2}\right)\left(\sin x - \frac{\sqrt{3}}{2}\right) = 0$.

34. $\left(\cos x - \frac{\sqrt{3}}{2}\right)\left(\cos x + \frac{\sqrt{3}}{2}\right) = 0$.

35. $2 \cos^2 x = 1 + \sin x$.

36. $2 \sin^2 x - 5 \cos x = 4$.

37. $\cot^2 x - 2 \csc x + 1 = 0$.

38. $2 \sin^2 x = 3 (1 - \cos x)$.

Group B

Solve and check each of the following equations for all positive values of the unknown angle less than 360°. (Use table on page 3 if necessary, and compute angles to the nearest degree.)

39. $6 \sin^2 \theta - 5 \sin \theta + 1 = 0$.

40. $8 \sin^2 x + 6 \cos x = 9$.

41. $4 \sin^2 \theta - 3 \sin \theta - 1 = 0$.

42. $\sin^2 y - \sin y = \cos^2 y$.

43. $\sec^2 x - 1 = \sqrt{3} \tan x$.

44. $\tan^2 x + 3 \cot^2 x = 4$.

Group C

Solve each of the following equations for all positive values of the unknown angle less than 360°:

45. $8 \sin^4 x - 6 \sin^2 x + 1 = 0$.

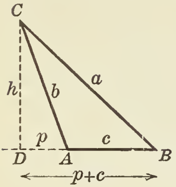
46. $\sec x \csc x - \cot x = \sqrt{3}$.

47. $4 \sin^2 x + 3 \csc^2 x = 7$.

48. $16 \sin^4 x - 16 \sin^2 x + 3 = 0$.

Law of cosines. On page 101 we established the law of cosines for an acute angle of a triangle. We shall now prove the law of cosines for an obtuse angle of a triangle.

Given $\angle A$ obtuse. Draw $CD \perp BA$ produced.



$$a^2 = h^2 + (p + c)^2.$$
$$a^2 = h^2 + p^2 + 2\,cp + c^2.$$
$$a^2 = (h^2 + p^2) + c^2 + 2\,cp.$$
$$a^2 = b^2 + c^2 + 2\,cp. \quad (h^2 + p^2 = b^2).$$

But, $\cos A = -\dfrac{p}{b}$. Hence $p = -b\cos A$.

$$\therefore\ a^2 = b^2 + c^2 - 2\,bc\cos A.$$

Similarly it can be shown that:

$$b^2 = c^2 + a^2 - 2\,ca\cos B.$$
$$c^2 = a^2 + b^2 - 2\,ab\cos C.$$

The law of cosines may be expressed as follows:

In any triangle, the square of any side equals the sum of the squares of the other two sides minus twice the product of these two sides and the cosine of their included angle.

Finding an angle of a triangle from the sides. We can find $\cos A$ in terms of the sides by using the law of cosines.

$$a^2 = b^2 + c^2 - 2\,bc\cos A.$$

Transposing, $\quad 2\,bc\cos A = b^2 + c^2 - a^2.$

Solving for $\cos A$, $\quad \cos A = \dfrac{b^2 + c^2 - a^2}{2\,bc}.$

In the same way we can derive formulas for $\cos B$ and $\cos C$.

$$\mathbf{a^2 = b^2 + c^2 - 2\,bc\cos A. \qquad \cos A = \frac{b^2 + c^2 - a^2}{2\,bc}.}$$

$$\mathbf{b^2 = c^2 + a^2 - 2\,ca\cos B. \qquad \cos B = \frac{c^2 + a^2 - b^2}{2\,ca}.}$$

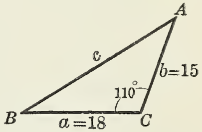
$$\mathbf{c^2 = a^2 + b^2 - 2\,ab\cos C. \qquad \cos C = \frac{a^2 + b^2 - c^2}{2\,ab}.}$$

Solving a triangle. In geometry, a triangle is determined (can be constructed) if three parts, at least one of which is a side, are given. In trigonometry, if the values of three parts, at least one of which is a side, are known, the values of the remaining three parts can be computed. The process of computing the unknown parts of a triangle from its given parts is called solving the triangle.

The law of cosines enables us to solve a triangle when we know two sides and their included angle or three sides.

Illustrative examples.

Example 1. Find the values of c, A, and B in the triangle shown.

Given	*To find*	
$a = 18.$ $b = 15.$ $C = 110°.$	c, A, and B.	

Solution

$$\begin{aligned} c^2 &= a^2 + b^2 - 2\,ab \cos C \\ &= 18^2 + 15^2 - 2(18)(15) \cos 110° \\ &= 324 + 225 - 540(-\ .3420) \\ &= 733.68. \\ c &= 27.0 = 27. \end{aligned}$$

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2\,bc} \\ &= \frac{15^2 + 27^2 - 18^2}{2(15)(27)} \\ &= \tfrac{630}{810} \\ &= .7777. \\ A &= 39°. \end{aligned}$$

$$\begin{aligned} \cos B &= \frac{c^2 + a^2 - b^2}{2\,ca} \\ &= \frac{27^2 + 18^2 - 15^2}{2(27)(18)} \\ &= \tfrac{828}{972} \\ &= .8518. \\ B &= 32°. \end{aligned}$$

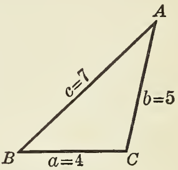
Check

$$A + B + C = 180°.$$
$$39° + 32° + 110° = 180°.$$
$$181° = 180°.$$

The results agree only approximately, since the answers found were only approximate, *i.e.*, angles to the nearest degree and the side to the nearest unit.

Example 2. Find the values of A, B, and C in the triangle shown.

Given	*To find*
$a = 4.$ $b = 5.$ $c = 7.$	A, B, and C.



Solution

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc} = \frac{25 + 49 - 16}{2(5)(7)} = .8285.$$
$$A = 34°.$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2\,ca} = \frac{49 + 16 - 25}{2(7)(4)} = .7142.$$
$$B = 44°.$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,ab} = \frac{16 + 25 - 49}{2(4)(5)} = -\ .2000.$$
$$C = 102°.$$

Check

$$A + B + C = 180°.$$
$$34° + 44° + 102° = 180°.$$
$$180° = 180°.$$

EXERCISES

In the following triangles, compute all unknown parts. Find the sides to the nearest unit and the angles to the nearest degree.

Given:

1. $a = 4$, $c = 7$, $B = 95°$.

2. $b = 12$, $c = 9$, $A = 110°$.

3. $a = 7$, $b = 10$, $C = 97°$.

4. $a = 5$, $b = 7$, $c = 10$.

5. $a = 13$, $b = 12$, $c = 5$.

6. $a = 6$, $b = 23$, $C = 172°$.

7. $a = 26$, $b = 23$, $c = 6$.

8. $a = 26$, $b = 20$, $c = 15$.

9. $a = 5$, $b = 8$, $c = 10$.

10. $b = 10$, $c = 7$, $A = 30°$.

11. $a = 26$, $b = 24$, $C = 10°$.

12. $a = 7$, $b = 24$, $c = 25$.

13. $b = 8$, $c = 14$, $A = 60°$.

14. $a = 12$, $b = 10$, $C = 91°$.

15. $b = 8$, $c = 11$, $A = 72°$.

16. $a = 24$, $b = 17$, $C = 37°$.

17. $a = 4$, $b = 5$, $c = 8$.

18. $a = 3$, $b = 4$, $c = 5$.

19. $a = 12$, $b = 10$, $c = 16$.

20. $a = 10$, $b = 10$, $c = 15$.

21. Two sides of a triangular lot are 14 yards and 17 yards in length and the angle between these two sides is 40°. Find the length of the third side.

22. A triangular plot of ground has two sides of lengths 10 rods and 12 rods and the angle included between these sides is 93°. Find the length of a fence surrounding the plot.

23. A and B are two points on opposite sides of a mountain. In order to find the horizontal distance between A and B, a third point C is taken from which A and B are visible, and the following measurements are made: $CA = 200$ feet, $BC = 300$ feet, $\angle BCA = 136°$. What is the horizontal distance between A and B?

24. The exact location of corners A and B of a triangular lot is known, but the stake that marked corner C has been lost. In the deed AB is given as 120 feet, BC 150 feet, and CA 180 feet. At what angle with AB should a surveyor run the line through A which will pass through corner C?

The Slow Rise of Trigonometry

Plane trigonometry developed very slowly. This was probably due to the fact that trigonometry in general was cultivated only as an aid to astronomical inquiry. Since what was known of plane trigonometry was of little use in such studies, meager attention was given it. It was not until 1551 that Rhaeticus defined the sine, cosine, and secant as functions of an angle, and named them *perpendiculum*, *basis*, and *hypotenusa*.

CUMULATIVE REVIEW

Chapters V, VI, and VII

1. Which of these statements are true? Which are false?

(*a*) The equation $2\sin^2 x - 3\sin x + 1 = 0$ is satisfied when $x = 90°$.

(*b*) The value of any function of an angle depends upon the value of the angle.

(*c*) The formula for the area of a triangle, $A = \frac{1}{2}bh$, is an identity.

(*d*) The relation between m and n as suggested in the table:

m	0	5	10	15	20	...
n	0	6	12	18	24	...

is expressed by the formula $6m = 5n$.

(*e*) If a taxicab fare is 25 cents for the first mile and 20¢ for each additional mile, the cost (C) for n miles is represented by the formula $C = 25 + 20(n - 1)$.

(*f*) The distance of the point $(-4, -3)$ from the origin is $+5$.

(*g*) The sine and cosecant of an angle agree in sign in any quadrant.

(*h*) If cot A is negative and A is obtuse, sin A is positive.

(*i*) If $\sin A = -\frac{7}{25}$, cos A must be negative.

(*j*) The six lines which represent geometrically the six functions of an angle in Q II, all lie in Q II.

2. Complete each of the following statements:

(*a*) If c is the hypotenuse of right triangle ABC, then $c^2 = a^2 + b^2 - 2ab \cos C$ becomes $c^2 = a^2 + b^2$ because the value of $\cos C$ is ? .

(*b*) If 2 parts of acid are mixed with 1 part of water, the solution will contain ? % of acid and ? % of water.

(*c*) If the area of a square is multiplied by 4, the perimeter of the square is multiplied by ? .

(*d*) In the formula $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$, if all letters stand for positive numbers only, then the letter which must have the least value is ? .

(*e*) The graph of a curve is observed to pass through points A, B, C, D, and E, whose readings are given in the table below:

POINT	A	B	C	D	E
y	4	1	0	1	4
x	-2	-1	0	1	2

The formula corresponding to this curve is ? .

(*f*) If A is obtuse and $\cos A = -\frac{24}{25}$, $\sin A =$? .

(*g*) The quadrant in which both the sine and the cosine increase as the angle increases is ? .

(*h*) The value of $\cos \frac{3\pi}{2}$ is ? .

(*i*) The value of $\tan 405° \cdot \cot\left(-\frac{\pi}{4}\right)$ is ? .

(*j*) The roots of the equation $\sin^2 x - \sin x = 0$ are 0°, 90°, and ? .

3. Express $\sec x$ and $\cos x$ in terms of $\tan x$.

4. A and B start from the same point and travel along roads that are perpendicular to each other. A travels 34 miles an

hour faster than B and at the end of 1 hour they are 50 miles apart. Find the rate of each.

5. In forwarding telegrams, a company charges r cents for the first 10 words and s cents for each additional word. Write the formula for the cost (C) of a telegram containing n words.

6. A circular pond is surrounded by a path 7 feet wide. If the area of the pond is 15,400 square feet, what is the area of the path? (Use $\pi = \frac{22}{7}$.)

7. Solve graphically: Two railroad stations are 180 miles apart. An express train traveling at a uniform rate of 30 miles an hour leaves one of the stations at the same time that a freight train traveling at the rate of 20 miles an hour leaves the other station. After how many hours will the trains pass each other?

8. Simplify the following expression in which x represents an acute angle:

$$\tan(270°+x)\cot(90°-x)$$
$$-\sin(180°-x)\cos(90°+x)-\cos(360°-x)\sin(270°-x).$$

9. Find and check two positive values of x less than 360° that satisfy the equation: $\tan x - 2\sin x = 0$.

10. The sides of a triangle measure 2 inches, 3 inches, and 4 inches respectively. Compute the three angles of the triangle correct to the nearest degree.

11. If $\sin A = \frac{5}{13}$ and $\cos A$ is negative, construct angle A and find the values of the other 5 functions of angle A.

12. To find the distance between two points A and B, which are separated by an impassable swamp, a civil engineer locates a third point C, accessible and visible from both A and B, and makes the following measurements: $AC = 200$ feet, $CB = 120$ feet, angle $ACB = 46°$. Find the distance AB.

CHAPTER VIII. GRAPHS AND FUNCTIONS

. . . *mathematics consists mainly in the study of functions and the study of functions is the study of the ways in which changes in one or more things produce changes in others.* — CASSIUS J. KEYSER.

THE MEANING OF DEPENDENCE

One of the chief aims of mathematics is to study changes in related quantities. In science, economics, business, and industry, we are frequently called upon to study how a change in one factor produces a change in one or more closely related factors. If we state these changes as equations, the study of the equations frequently leads to discoveries of the laws of nature and to the improvement of the conditions of life about us.

You are already acquainted with the fact that a change in the radius of a circle makes a related change in its circumference. This *dependence* of the circumference upon the radius is expressed by the equation $C = 2\pi r$. Whenever the value of r changes, the corresponding value of C also changes. Hence r and C are said to be *variables*, and the number symbol π, which maintains the same value throughout the problem or discussion, is called a *constant*.

We observe that the value of C is dependent upon the value assigned to r. Therefore since r may have any value, it is called the *independent variable* and C is called the *dependent variable.*

If, on the other hand, we regard C as the variable to which we assign values, r is dependent on C. In this case, C is the independent variable and r the dependent variable.

Another way of stating that the circumference of a circle depends upon the radius is that the circumference of a circle is a *function* of the radius. It is equally correct to say that the radius is a function of the circumference.

EXERCISES

Read the first two statements carefully, then complete the remaining statements.

1. The cost of tea increases with the amount purchased. Therefore the *cost* is a function of the *amount purchased.*

2. The value of the cosine of an angle decreases as the angle increases. Hence the *cosine* is a function of the *angle.*

3. The velocity at which an object falls from rest increases with the number of seconds it falls. Hence the ? is a function of the ? .

4. The value of the cotangent of an angle decreases as the angle increases. Hence the ? is a function of the ? .

5. The volume of a gas decreases as the pressure increases. Hence the ? is a function of the ? .

6. The amount of postage on a letter depends upon the weight of the letter. Therefore the ? is a function of the ? .

7. The value of the sine of an angle depends upon the size of the angle. Hence the ? is a function of the ? .

8. The electrical resistance in a wire varies as the length of the wire. Hence the ? is a function of the ? .

9. The value of the tangent of an angle varies as the angle. Hence the ? is a function of the ? .

10. The cost of a railroad ticket depends upon the distance to be traveled. Hence the ? is a function of the ? .

11. In the formula $C = \frac{5}{9}(F - 32)$, C is a function of ? .

12. In the formula $A = b \times h$, A is a function of ? and ? .

13. In the formula $V = 2\pi r^2 h$, V is a function of ? and ? .

14. If $y = x + 2$, y is a function of ? .

Equations of the First Degree — Linear Equations

Recall fact 45. *To represent a linear equation as a graph:*

1. *Find corresponding pairs of values of the variables.*

2. *Each pair of values of the variables determines a point with reference to the axes. Plot three or more of these points and join them with a straight line.*

Illustrative examples.

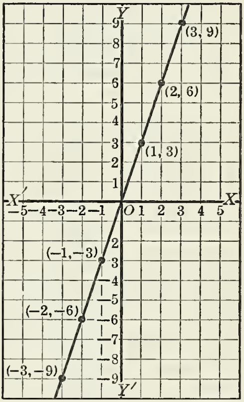
Example 1. Graph $y = 3x$.

Analysis

Assign to x the values $-3, -2, -1, 0, 1, 2, 3$ and substitute each of them in $y = 3x$ to get corresponding values of y.

Solution

If $x =$	Then $y =$
-3	-9
-2	-6
-1	-3
0	0
1	3
2	6
3	9



Plotting each of these points and drawing a line through them, we obtain the graph as in the adjoining diagram.

The graph of every linear equation is a straight line. Thus the name *line*ar equation.

Something to think about.

1. Since the graph of every linear equation is a straight line, how many points is it necessary to plot to obtain the line?

2. What change takes place in y as x increases?

3. Is y a function of x? Why?

4. How many pairs of values are there that satisfy the equation?

Example 2. Draw the graph of $3y = 2x + 6$.

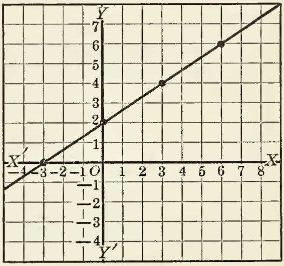
Analysis

In order to make a table of corresponding values, it is necessary to solve the equation for y in terms of x.

Solution

$$3y = 2x + 6.$$
$$y = \tfrac{2}{3}x + 2.$$

If $x =$	Then $y =$
-3	0
0	2
3	4
6	6



Something to think about.

1. Did you know beforehand that the graph of $3y = 2x + 6$ would be a straight line? Why?

2. Suggest a reason why the values -3, 0, 3, and 6 were assigned to x rather than such values as -2, 1, or $+2$.

3. What is the value of y when $x = 0$?

4. What is the value of x when $y = 0$?

5. Since a change takes place in y as x increases, is y a function of x?

EXERCISES

1. (*a*) Draw the graph of $y = x$. Assign to x the values: $-3, -2, -1, 0, 1, 2, 3$.

(*b*) What angle does the graph make with the x-axis?

(*c*) What is the tangent of the angle found in (*b*)?

2. Using the same axes, draw the graphs of:

(*a*) $y = 2x$. (*c*) $y = -x$.

(*b*) $y = 4x$. (*d*) $y = -3x$.

3. Using the same axes, draw the graphs of:

(*a*) $y = \frac{3}{4}x$. (*c*) $y = -\frac{1}{2}x$.

(*b*) $y = \frac{2}{3}x$. (*d*) $y = -\frac{2}{3}x$.

4. In the equation $y = bx$, how does a change in b, the coefficient of x, affect the direction of the graph? What quadrants does the graph cross when the coefficient of x is positive? When it is negative? If this coefficient is increased gradually, how does the line move?

Draw the graph of each of the following equations:

5. $y = x + 2$.

6. $y = x - 5$.

7. $y = 3x + 2$.

8. $3x - y = 5$.

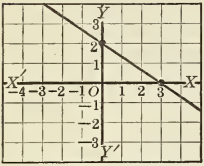
9. $2y - x + 4 = 0$.

10. $3y - 2x + 6 = 0$.

The intercept method of graphing a linear equation. Since the graph of a linear equation is a straight line, only two points are needed to determine the graph. Of course, the two points should be far enough apart to warrant the accurate drawing of the straight line. There are two points which can be easily obtained from a linear equation; these are points where the graph crosses (intercepts) the x- and y-axes. These points are called the *intercepts*. Since $x = 0$ for any point on the y-axis, the intercept on the y-axis is found by letting $x = 0$ in the equation and solving for y. Similarly, the intercept on the x-axis is found by letting $y = 0$.

Illustrative example. Graph the equation $2x + 3y = 6$.

Solution



$$\text{Substituting } x = 0.$$
$$2 \cdot 0 + 3y = 6.$$
$$y = 2.$$
$$\text{Substituting } y = 0.$$
$$2x + 3 \cdot 0 = 6.$$
$$x = 3.$$

Plotting these points and joining them with a straight line, we have the graph above.

Something to think about.

1. Could the intercept method be used to determine the graph of $y = 2x$? Why?

2. If a pair of values for x and y, other than the intercepts, are found from the equation, explain how the point represented by these values may be used to check the graph.

EXERCISES

Using the intercept method, draw the graph of each of the following equations.

1. $x + y = 6$.

2. $x - y = 2$.

3. $2x + y = 4$.

4. $x + 2y = 6$.

5. $2x + 5y = 10$.

6. $4x - 3y = 12$.

7. $y = 3x + 3$.

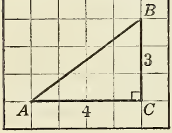
8. $y = 3x - 6$.

9. $y = 4x - 8$.

10. $2x + 3y = 9$.

11. $3x - 2y = 7$.

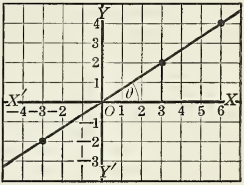
12. $4x + 6y = 15$.



The slope of a line. In the adjoining diagram,
$$\tan A = \frac{BC}{AC} = \frac{3}{4}.$$

The line AB rises 3 units in a horizontal distance of 4 units. The ratio $\frac{3}{4}$ is called the *slope* of the line. When a line, such as AB, rises to the right, its slope is positive because the tangent of an acute angle is positive; when a line rises to the left, the slope is negative because the tangent of an obtuse angle is negative.

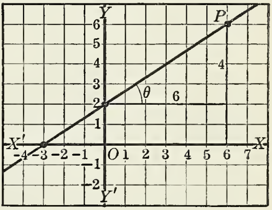
Finding the slope of a straight line from its equation. Let us consider the graph of $3y = 2x$ as shown in the adjoining diagram. From the graph it can be seen that the slope of the line is $\frac{2}{3}$. Since the slope of the line is $\frac{2}{3}$, every point on the line is such that the ratio



$$\frac{\text{ordinate}}{\text{abscissa}} = \frac{2}{3} = \tan\theta.$$

But if we should solve the given equation $3y = 2x$ for y in terms of x, we have $y = \frac{2}{3}x$. In this equation you will observe that the coefficient of x, $\frac{2}{3}$, is the slope ($\tan\theta$) of the line. Note that this relation between the coefficient of x and the slope of the line is true only when the equation has been solved for y and is in the form $y = mx$.

Now let us consider the graph of $3y = 2x + 6$ as shown in the adjoining diagram. From the graph you can see that the slope is $+\frac{2}{3}$ (ratio of opposite to adjacent side, and positive because it rises to the right) and that the y-intercept is $+2$.



Solving $3y = 2x + 6$ for y, we have $y = \frac{2}{3}x + 2$.

Comparing this equation with facts just found, we learn that:

(1) the coefficient of x, $\frac{2}{3}$, is the slope ($\tan\theta$).

(2) the constant term $(+2)$ is the y-intercept.

In general then, in an equation of the form $y = mx + b$, the coefficient of x (m) is the slope and the y-intercept is the constant term (b).

Something to think about.

1. What is the slope of the graph of the equation $y = x + 4$?

2. What is the slope of the graph of the equation $y = 4$? How would you graph this equation?

3. Where is the graph of the equation $x = 3$? $x = 2$? $x = 1$? $x = 0$? $y = 10$?

4. What is the slope of $2x + y = 4$? of $2x + y = 3$? What conclusion can you draw from a comparison of the slopes of these lines?

5. If a graph passes through the origin, what must be the value of the constant term in its equation?

EXERCISES

Determine the slope and the y-intercept of the graph of each of these equations:

1. $y = x + 3.$

2. $y = x - 4.$

3. $y = -x + 2.$

4. $y = 3x + 5.$

5. $2x - 3y = 6.$

6. $2y = x - 1.$

7. $2x + y = 0.$

8. $x - 2y = 0.$

9. $y = 3x.$

10. $5y = 2x.$

11. $3y = 2x + 4.$

12. $y = -2x - 2.$

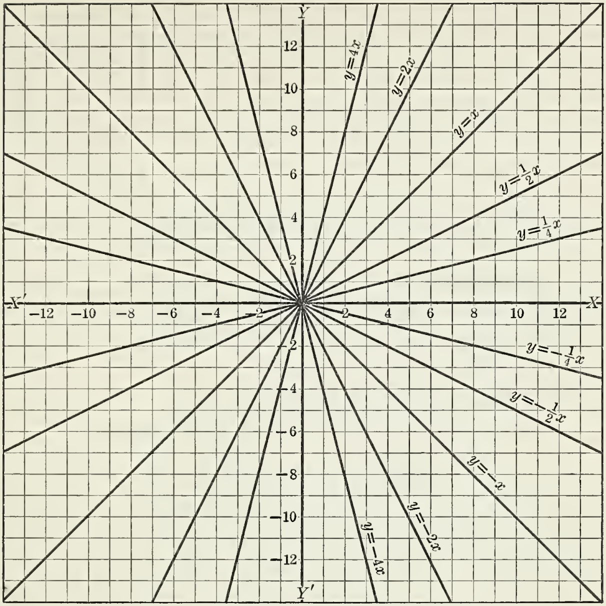
13. $2x - y = 0.$

14. $3x - 4y = 7.$

Something to think about. In each of the equations of the last exercise, how could you find the number of degrees in the angle formed by the graph of the equation and the x-axis?

Families of linear equations.

A. Type $y = mx$. If, in the equation $y = mx$, different values, positive and negative, are assigned to m, a *family* of graphs can be drawn as shown in the diagram. Some very interesting and instructive conclusions may be derived from a study of this figure.



1. All the graphs are straight lines. Why?

2. All the graphs pass through the origin. Under what condition does the graph of an equation pass through the origin?

3. As the value of m increases, the graph of $y = mx$ appears to rotate in a counter-clockwise direction. As the value of m decreases, the graph appears to rotate in a clockwise direction.

B. Type $y = mx + b$. Are the graphs of equations of this type straight lines? Tell why? Do the graphs pass through the origin? Tell why?

Let us now see the effect on this family of graphs when (1) m changes and b remains constant; (2) when b changes and m remains constant.

(1) m changes, b remains constant.

Plot on the same axes the graphs of:

(*a*) $y = x + 2$. (*c*) $y = 3x + 2$.
(*b*) $y = 2x + 2$. (*d*) $y = 4x + 2$.

Through what point do all of these graphs pass? What are the coördinates of this point? How could you have foretold these facts from the equations?

What are the y-intercepts? What causes this family of graphs to appear to rotate about the point $(0, 2)$?

(2) b changes, m remains constant.

Plot on the same axes the graphs of:

(*a*) $y = 2x + 1$. (*c*) $y = 2x + 3$.
(*b*) $y = 2x + 2$. (*d*) $y = 2x + 4$.

Do these lines appear to have the same slope? How could you tell from the equations whether they have or have not the same slope?

What are the y-intercepts? What causes this family of graphs to appear to be *translated* (that is, moved so as to remain parallel to one another)?

EXERCISES

1. Graph the first of the following equations and obtain the graphs of (*b*), (*c*) and (*d*) by translation.

(*a*) $y = 3x - 1$. (*c*) $y = 3x + 4$.
(*b*) $y = 3x + 2$. (*d*) $y = 3x + 7$.

2. Draw the graph of the equation $y = 2x$. Using this graph, draw the graphs of the following equations without plotting.

(*a*) $y = 2x - 1$. (*b*) $y = 2x + 5$.

3. Write the equation of each of the lines whose slope and y-intercept are as follows. (The equations are of the form $y = mx + b$.)

Slope	3	2	− 2	− 1	0	− 3	5	$\frac{1}{3}$
y-intercept	2	3	4	3	4	− 1	0	0

4. First determine the slope and then write the equation of the line passing through the point P (2, 3) and having its y-intercept 1.

5. Write the equation of the line passing through the point P (3, 4) and parallel to the line $y = x + 4$. (The equation is of the form $y = mx + b$. The slopes of the two lines are the same.)

6. Write the equation of the line passing through the point (0, 0) and making an angle of 45° with the x-axis.

A Study in Variation

Problem. The amount of gasoline used by an automobile varies directly as the number of miles traveled. If an automobile uses 4 gallons in going 56 miles, how much will it use in going 168 miles?

In order to solve problems such as this one, it will first be necessary to learn the meaning and use of the phrase "varies directly as."

The meaning of direct variation. Let us consider the equation $y = 3x$. From this equation we know that the ratio $\frac{y}{x}$ is always equal to 3. When two variables, such as y and x in the equation $y = 3x$, are so related that their ratio is constant, either one of them is said to *vary directly as* the other. Often the word "directly" is omitted, for "varies as" means the same as "varies directly as."

Thus, in general, if $\frac{y}{x} = K$, or $y = Kx$, and K is a constant, then y varies directly as x.

Illustrative examples.

Example 1. Express the following statement in the form of an equation:

The cost of coffee varies directly as the price.

Solution

If c represents the cost of coffee and p the price, then $\frac{c}{p} = K$, or $c = Kp$.

Example 2. Express the following statement in the form of an equation:

The circumference of a circle varies as the diameter.

Solution

If C represents the circumference and d the diameter, then C varies as d, from which it follows that $\frac{C}{d} = K$, or $C = Kd$.

EXERCISES

Express each of the following statements in the form of an equation:

1. The cost (C) of tea varies directly as the number (N) of pounds bought.

2. The distance (D) a train travels varies directly as the rate (R).

3. The velocity (V) at which an object falls from rest varies directly as the number (N) of seconds it falls.

4. The amount (A) of postage on a letter varies as the weight (W) of the letter.

5. The cost (C) of a taxi fare varies as the distance (d) traveled.

6. The pressure (P) in pounds per square inch of a column of water varies directly as the height (h) of the column in feet.

Another way of expressing direct variation. Instead of saying that y varies directly as x, we can say that y is *proportional* to x.

But, you say, a proportion has four terms. Why, then, in $y = Kx$, in which there are only two variables, do we say "y is proportional to x"? Have we really a proportion here? Let us examine the table of values for $y = 3\,x$.

If $x =$	1	2	3	4	5	6	7	8
Then $y =$	3	6	9	12	15	18	21	24

Select any two values of x and the corresponding values of y. Do they form a true proportion? For example, are these proportions true: $\frac{1}{3} = \frac{4}{12}$? $\frac{3}{9} = \frac{7}{21}$? Can you find two pairs of corresponding values which do not make a proportion? Is the continued proportion $\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12} \cdots$ true?

That these numbers are proportional in general can be shown as follows: If y becomes y_1 when x becomes x_1, and y_2 when x becomes x_2, then $y_1 = kx_1$ and $y_2 = kx_2$.

Dividing, we have $$\frac{y_1}{y_2} = \frac{\cancel{k}x_1}{\cancel{k}x_2} = \frac{x_1}{x_2}.$$

EXERCISES

Write the equation for each of the following statements, and use two pairs of values to show that you can make a true proportion.

1. The distance a car will travel in 5 hours is proportional to its speed.

2. The simple interest on \$200 at 6% is proportional to the number of years the money is invested.

3. The cost of a bag of sugar is proportional to the number of pounds in the bag.

4. If a train travels 40 miles an hour, the distance it will go is proportional to the time it runs.

5. The amount Mrs. Williams must pay for two-cent stamps is proportional to the number she buys.

6. If I invest money at 5%, my annual income will be proportional to the amount invested.

By selecting values from the table on page 3, demonstrate your answers to the following questions:

7. Is the sine of an angle proportional to the angle?

8. Is the sine of an angle proportional to the tangent of the angle?

Illustrative example. The amount of gasoline used by an automobile varies directly as the number of miles traveled. If an automobile uses 4 gallons in going 56 miles, how much gasoline will it use in going 168 miles?

Analysis

From the statement that the amount of gasoline (G) varies directly as the number of miles (N) we can write the equation $G = KN$, where K is a constant. Why? By using the values given in the problem ($G = 4$, $N = 56$), we can find the numerical value of K, which will give us a formula to use in solving the problem.

Solution

Substituting $G = 4$, $N = 56$ in $G = KN$, $\quad 4 = 56K$.

$$K = \tfrac{1}{14}.$$

Hence, $\quad G = \tfrac{1}{14}N$.

But the problem requires us to find G when $N = 168$.

$$G = \tfrac{1}{14}N.$$

$$G = \tfrac{1}{14}(168) = 12.$$

The automobile will use 12 gallons of gasoline to go 168 miles.

In general, then, problems in variation are solved by (1) setting up the equation which shows the variation, (2) substituting given values to find the constant, and (3) using the resulting equation as a formula to solve the problem.

EXERCISES

1. The velocity (V) of a body falling from rest varies directly as the time (t) it falls. If $V = 160$ feet per second after 5 seconds of falling, find the velocity at the end of 8 seconds.

2. The weight of a liquid is directly proportional to (varies directly as) the volume. If 100 cubic feet of water weighs 6250 pounds, what is the weight of 125 cubic feet?

3. The weight of steel wire is directly proportional to its length. Find the weight of 130 feet of wire, if 100 feet of the same kind weighs 30 pounds.

4. The price of a railroad ticket is directly proportional to the distance. Find the price of a ticket to a city 50 miles away, if the price of a 35-mile trip is \$1.19?

5. The circumference of a circle varies directly as the radius. If the circumference of a circle of radius 7 inches is 44 inches, what is the circumference of a circle of radius 17 inches?

6. The perimeter of a square varies directly as the length of one side. If the perimeter of a square whose side is 4.6 inches is 18.4 inches, what is the perimeter of a square whose side is 2.9 inches?

7. If y varies directly as x and $x = 3$ when $y = 9$, what is the value of y when $x = 28$?

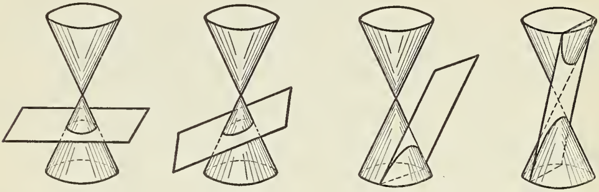
8. If x varies directly as y, and $x = 24$ when $y = 3$, what is the value of x when $y = 5$?

9. If r varies directly as s, and $r = 5$ when $s = 25$, what is the value of r when $s = 16$?

10. If k is directly proportional to z, and $k = 6$ when $z = 1.5$, what is the value of k when $z = 3.9$?

Equations of the Second Degree — Conic Sections

About 225 B.C., Apollonius of Perga, sometimes called the "Great Geometer," wrote a treatise on "Conic Sections." By conic sections is meant the curves of intersection which result when a circular double cone is cut by a plane, as shown below.



(a) (b) (c) (d)

1. If the cutting plane is parallel to the base of the cone, the curve of intersection is a *circle* (a).

2. If the cutting plane is inclined to the bases of the cone, and yet does not pass through either base, the curve of intersection is an *ellipse* (b).

3. If the cutting plane is parallel to a line connecting the vertex of the cone with any point in the perimeter of its base, the curve of intersection is a *parabola* (c).

4. If the cutting plane is inclined to the bases and cuts both bases, the curve of intersection is a *hyperbola* (two branches)(d).

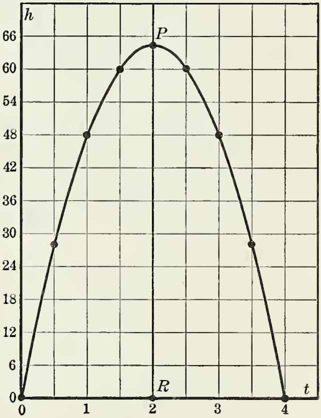
The work of the Greek mathematicians, however, was strictly geometrical and it was not until the 17th century that the curves of intersection of the cone were investigated from the algebraic point of view. Long afterwards the practical importance of the conic sections began to be emphasized, and in our present day we find that large numbers of scientific problems in astronomy, physics, and architecture depend for solution upon a knowledge of these curves and their properties.

Each of these four conic sections may be represented by a general equation.

The parabola. The parabola is a curve that can be used to represent phenomena in astronomy, physics, engineering, and the like. The path of any projectile, whether it be a baseball or a cannon ball, is a parabola if the resistance of the air is neglected. Many arches and cables of bridges are in the form of a parabola. Automobile headlights, searchlights, and reflecting telescopes are usually parabolic in form. A brief study of the problem given below will help you to learn more about the parabola.

If a bullet is fired from a rifle with a given velocity at a given angle to the ground, the height (h) in feet reached after t seconds may be expressed by the formula $h = 64\,t - 16\,t^2$. The graph of this equation, using the values found in the table, is given in the figure below. From this graph find the maximum height to which the bullet rises and the number of seconds required for it to reach the ground.

If $t =$	Then $h =$
0	0
$\frac{1}{2}$	28
1	48
$1\frac{1}{2}$	60
2	64
$2\frac{1}{2}$	60
3	48
$3\frac{1}{2}$	28
4	0
$4\frac{1}{2}$	-36



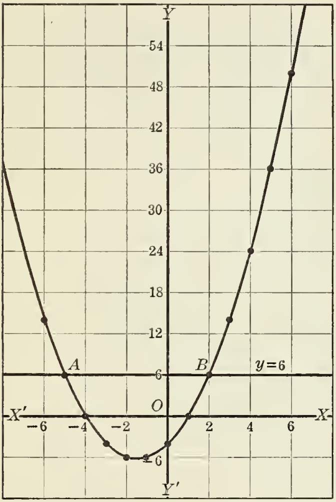
The following observations may be made from the graph:

1. The graph is not a straight line but is curved.

2. As t changes from 0 to 2, the value of h increases and reaches its maximum (greatest) value at P, where it is 64.

3. For every point on the curve to the right of line PR there is a corresponding point on the curve to the left. Hence the curve is *symmetrical* about the line PR and PR is called the axis of symmetry of the curve.

This curve is the graph of the quadratic equation $h = 64\,t - 16\,t^2$, which is of the form $y = ax^2 + bx + c$. Such a curve, as you know, is called a parabola. The graph of every quadratic equation of the form $y = ax^2 + bx + c$ is a parabola.



Solving quadratic equations graphically.

Illustrative example. Graph the quadratic equation

$$y = x^2 + 3\,x - 4.$$

Solution

Let x have values from $-\,6$ to $+\,6$ and find the corresponding values of y. These values may be arranged as in the following table:

If $x =$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
Then $x^2 + 3\,x - 4$ or $y =$	14	6	0	-4	-6	-6	-4	0	6	14	24	36	50

By using the values in the table, the curve in the diagram above may be drawn.

Something to think about.

1. As expected, the curve is a ? .

2. The curve cuts the x-axis at ? and ? .

3. The value of y at each of the two points where the curve cuts the x-axis is ? . If we should substitute 0 for y in the equation $y = x^2 + 3x - 4$, we get $0 = x^2 + 3x - 4$. Solve this equation by factoring. What are the two roots? How do they compare with the values of x at the points where the curve cuts the x-axis?

Therefore given the function $y = x^2 + 3x - 4$ from which it is required to solve the equation $x^2 + 3x - 4 = 0$, we graph the given function and then determine the abscissas where the curve crosses the x-axis, *i.e.*, the line $y = 0$.

If the roots of the equation $x^2 + 3x - 4 = 6$ be required, we graph the function $y = x^2 + 3x - 4$ and the line $y = 6$. The abscissas of the points of intersection (points A and B; $x = -5$ and $x = +2$ in the diagram) of the curve and the line will give the desired roots. These roots may be checked by substitution in $6 = x^2 + 3x - 4$, or by solving the quadratic equation $x^2 + 3x - 4 = 6$.

The roots of a quadratic equation are not always integers; they may be fractions or decimals. In such cases we find the roots "approximately," generally to the nearest tenth.

EXERCISES

1. Plot the graph of:

(a) $y = x^2 + 5x + 6$.
(b) $y = x^2 - 3x + 2$.
(c) $y = x^2 + 3x - 4$.
(d) $y = -x^2 + x + 2$.
(e) $y = 2x^2 - x - 5$.
(f) $y = x^2 - 3$.

2. From the graph of each of the following equations determine the values of x when $y = 0$:

(a) $y = x^2 + 3x + 2$.
(b) $y = x^2 - 9$.
(c) $y = -x^2 + 5x - 6$.
(d) $y = x^2 - 2x + 1$.

3. Plot the graph of $y = x^2 - 4x + 3$ and from the graph determine the roots of $x^2 - 4x + 3 = 0$.

4. Solve graphically:

(*a*) $x^2 - 5x + 4 = 0$. (*c*) $x^2 - 3x = 10$.

(*b*) $2x^2 - x - 3 = 0$. (*d*) $x^2 - 4 = 0$.

5. Find the roots graphically, correct to the nearest tenth:

(*a*) $x^2 + 2x - 1 = 0$. (*c*) $x^2 - 8x = 15$.

(*b*) $x^2 - 3x - 2 = 0$. (*d*) $x^2 - 6x = 10$.

6. Draw the graph of $y = x^2 - 5x + 6$, and from it:

(*a*) Find the roots of $x^2 - 5x + 6 = 0$.

(*b*) Determine the roots of: $x^2 - 5x + 6 = 3$, $x^2 - 5x + 6 = +1$, $x^2 - 5x = 0$, $x^2 - 5x = 4$.

7. Draw the graph of the function $x^2 - 4x$, and from it:

(*a*) Determine the roots of $x^2 - 4x = 0$.

(*b*) Determine the approximate roots of: $x^2 - 4x = 3$, $x^2 - 4x = 1$, $x^2 - 4x = -1$, $x^2 - 4x - 4 = 0$.

8. Draw the graph of the function $x^2 + 3x$, and from it determine the roots of: $x^2 + 3x = 0$, $x^2 + 3x + 2 = 0$, $x^2 + 3x = 1$, $x^2 + 3x = 2$.

9. Draw the graph of the function $x - x^2$, and from it determine the roots of: $x - x^2 = 0$, $x - x^2 = -2$.

10. (*a*) Draw the graph of $x = y^2$. (Assign values to y and find the corresponding values of x.)

(*b*) On the same axes draw the graph of $x = -y^2$.

(*c*) The graph of each equation is symmetrical with respect to the ? axis.

11. The area of a square in terms of a side is $K = s^2$.

(*a*) Draw the graph of $K = s^2$.

(*b*) If the value of s increases, does the value of K increase?

(*c*) From the graph determine which increases more rapidly, s or K?

(*d*) From the graph determine the side of a square whose area is 10; 25; 30; 49; 75.

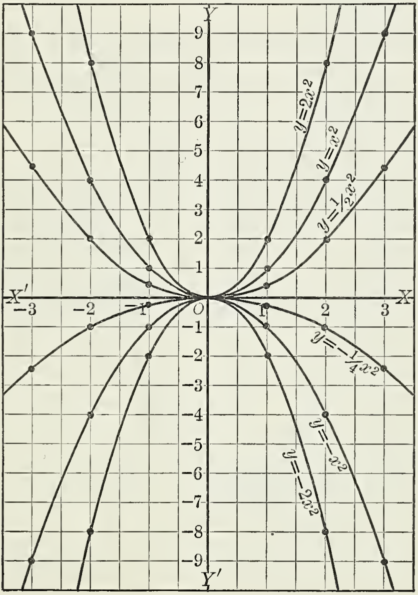
12. The area of a circle in terms of its radius r is $K = \frac{22}{7} r^2$. Draw the graph of $K = \frac{22}{7} r^2$ and from it:

(*a*) Determine the area of a circle whose radius is $3\frac{1}{2}$; $1\frac{3}{4}$; 7; 2.

(*b*) Determine the radius of a circle whose area is 22; 33; 10; 50.

Families of parabolas.

A. Type $y = ax^2$. If, in the equation $y = ax^2$, different values, positive and negative, are assigned to a, a family of graphs can be drawn as shown in the diagram.



Some very interesting and instructive conclusions may be derived from a study of the diagram on page 227.

1. Do all these parabolas pass through the same point? Could you have foretold this fact from the equations?

2. With respect to which axis is each curve symmetrical?

3. Does the parabola turn up or down when a (the coefficient of x^2) is positive? negative?

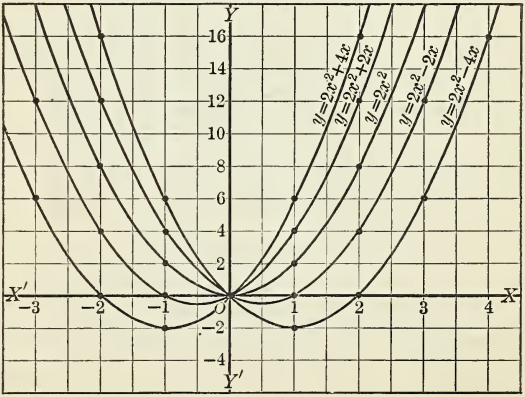
4. With respect to which axis are the graphs of $y = ax^2$ and $y = - ax^2$ taken together symmetrical?

5. Describe how the curve $y = ax^2$ changes (a) when a increases; (b) when a is negative; (c) when a approaches zero; (d) when a becomes zero; (e) when a becomes positive; (f) when a becomes very large.

Something to think about. Imagine you have drawn the graphs of $x = ay^2$ and $x = - ay^2$ and picture their relative positions in your mind. Is the graph of each of these equations symmetrical to any line?

B. Type $y = ax^2 + bx$, as b changes.

Below is a graph of functions of the type $y = ax^2 + bx$.



Observations from the graph.

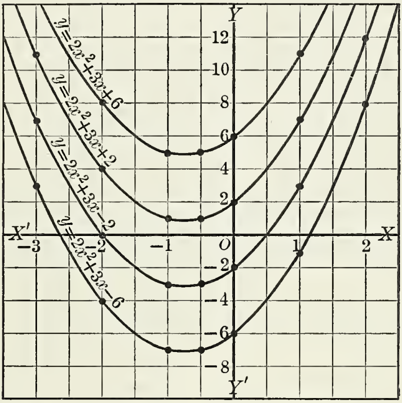
1. Do all these parabolas pass through a common point? How could you have discovered this fact without drawing the graphs of these equations?

2. The axis of symmetry in each case will be a line parallel to the y-axis, passing through the lowest point of each curve, called "the turning point." Why?

C. Type $y = ax^2 + bx + c$, as c changes.

Consider the graphs of the equations:

(1) $y = 2x^2 + 3x + 6$. (3) $y = 2x^2 + 3x - 2$.
(2) $y = 2x^2 + 3x + 2$. (4) $y = 2x^2 + 3x - 6$.



Observations from the graph.

1. Observe that the graphs show that the equations differ only in the constant term. In which direction is the curve translated (shifted) when c is increased by 4? When c is decreased by 8?

2. What conclusion can you draw as to the changes which take place in the graph as the constant term increases or decreases?

EXERCISES

1. Without actually constructing a table, sketch the graph of each of the following equations, describe its position and axis of symmetry.

(*a*) $y = x^2$.
(*b*) $y = x^2 + 1$.
(*c*) $y = -x^2$.
(*d*) $y = -x^2 + 1$.
(*e*) $x = y^2$.
(*f*) $x = y^2 + 1$.
(*g*) $x = -y^2$.
(*h*) $x = -y^2 + 1$.
(*i*) $y = 2x^2 + 1$.
(*j*) $y = -3x^2 + 7$.
(*k*) $y = -x^2 + 2x - 1$.
(*l*) $y = 3x^2 - 10x + 3$.

2. Sketch the graphs of $y = x^2 - 3x$ and $y = -x^2 + 3x$. How do they differ?

3. Draw the graph of each of the following equations and state what effect an increase in the value of the constant term c has on the graph.

(*a*) $y = x^2 + x - 2$.
(*b*) $y = x^2 + x - 1$.
(*c*) $y = x^2 + x$.
(*d*) $y = x^2 + x + 1$.
(*e*) $y = x^2 + x + 2$.

4. Tell by inspection which of the graphs of these equations are symmetrical to the x-axis; which to the y-axis.

(*a*) $y = 3x^2$.
(*b*) $y^2 = 10x$.
(*c*) $y^2 = 3x + 4$.
(*d*) $y = -2x^2$.
(*e*) $y = -3x^2 + 1$.
(*f*) $y = 10x - 16x^2$.
(*g*) $x = 3y^2 + 2$.
(*h*) $x = -3y^2 - 5$.
(*i*) $x = 3y^2$.
(*j*) $x = 3y^2 - 10y$.
(*k*) $x = 3y^2 - 10y + 3$.
(*l*) $x = -3y^2 + 10y - 3$.
(*m*) $y = x^2 - 6x + 9$.
(*n*) $x^2 + y = 3$.
(*o*) $x + y^2 = 5$.
(*p*) $x^2 - 2y = 0$.

5. Describe the changes which take place in the graph of the equation:

(*a*) $y = ax^2$ as a increases from -1 to $+3$.
(*b*) $y = 2x^2 + 5x + c$ as c increases from -4 to $+4$.

Solving quadratic equations algebraically. Probably you have observed that in solving equations the graphic method gave results correct to the nearest tenth only. Therefore it is necessary for us to learn methods which will yield more accurate results. You have already learned how to apply your knowledge of factoring in solving quadratic equations. Explain how this is done and solve the following equations, by this method:

(*a*) $x^2 + 4x + 4 = 0.$
(*b*) $x^2 - 7x + 12 = 0.$
(*c*) $x^2 + 9x - 10 = 0.$
(*d*) $y^2 + 17y - 60 = 0.$
(*e*) $y^2 + y - 2 = 0.$
(*f*) $y^2 + 7y + 10 = 0.$
(*g*) $3y^2 + 22y + 7 = 0.$
(*h*) $5x^2 - 9x + 4 = 0.$
(*i*) $17x^2 + 16x - 1 = 0.$
(*j*) $x^2 = 15x - 56.$
(*k*) $3x = 5x^2 - 2.$
(*l*) $15 = x^2 - 2x.$

Solution by completing the square. The factoring method is very easily applied provided the quadratic expression is factorable. Unfortunately this is not frequently the case in practical problems and so we must consider a new method called "completing the square." This method is particularly important because it leads to a formula by means of which any quadratic equation may be solved rapidly.

The following exercises will help you. What terms should be added in order to make the following expressions perfect trinomial squares?

(*a*) $x^2 + 2x + ?$.
(*b*) $x^2 + 4x + ?$.
(*c*) $x^2 + 6x + ?$.
(*d*) $x^2 + 8x + ?$.
(*e*) $x^2 - 8x + ?$.
(*f*) $x^2 - 10x + ?$.
(*g*) $x^2 - 13x + ?$.
(*h*) $x^2 - 15x + ?$.
(*i*) $x^2 - 9x + ?$.
(*j*) $x^2 + 7x + ?$.
(*k*) $x^2 + 5x + ?$.
(*l*) $x^2 - x + ?$.

How is the term to be added in the examples above obtained from the coefficient of x?

Illustrative examples.

Example 1. Solve the equation $2x^2 = -8x + 10$.

Method	*Solution*
1. Change the given equation to the form of $ax^2 + bx = c$.	$2x^2 + 8x = 10.$
2. If a is not 1, divide both sides of the equation by the coefficient of x^2.	$x^2 + 4x = 5.$
3. Now take one-half the coefficient of x; square it; and add this quantity to both sides of the equation.	$x^2 + 4x + 4 = 5 + 4.$
4. Write the left side as the square of a binomial and combine the terms on the right side.	$(x+2)^2 = 9.$
5. Take the square root of each side, using the $\pm$ sign before the square root of the right side.	$x + 2 = \pm 3.$
6. Solve the resulting two linear equations.	$x+2=+3$, $x=1$ | $x+2=-3$, $x=-5$
7. Check the resulting roots.	

Check	*Check*
$2x^2 = -8x + 10$	$2x^2 = -8x + 10$
$2 = -8 + 10$	$50 = +40 + 10$
$2 = 2$	$50 = 50$

Example 2. Solve correct to nearest tenth $3x^2 - 5x = 7$.

Solution

$$3x^2 - 5x = 7.$$
$$x^2 - \tfrac{5}{3}x = \tfrac{7}{3}.$$
$$x^2 - \tfrac{5}{3}x + \tfrac{25}{36} = \tfrac{7}{3} + \tfrac{25}{36}. \quad [\tfrac{1}{2} \text{ of } \tfrac{5}{3} = \tfrac{5}{6};\ (\tfrac{5}{6})^2 = \tfrac{25}{36}.]$$
$$(x - \tfrac{5}{6})^2 = \tfrac{109}{36}.$$
$$x - \tfrac{5}{6} = \pm\sqrt{\tfrac{109}{36}} = \pm\tfrac{1}{6}\sqrt{109}.$$
$$x = \tfrac{5}{6} \pm \tfrac{1}{6}\sqrt{109}.$$

$x = \frac{5}{6} + \frac{1}{6}\sqrt{109}$	$x = \frac{5}{6} - \frac{1}{6}\sqrt{109}$
$= .83 + (\frac{1}{6} \times 10.44)$	$= .83 - (\frac{1}{6} \times 10.44)$
$= .83 + 1.74$	$= .83 - 1.74$
$= 2.57.$	$= -\ .91.$
$\therefore\ x = 2.6.$	$x = -\ .9.$

EXERCISES

Group A

Solve each of the following by completing the square.

1. $x^2 - 4x + 3 = 0.$
2. $x^2 - 4x = 21.$
3. $3x^2 = 2x + 8.$
4. $2a^2 + 5a + 2 = 0.$
5. $3m^2 + 4m = 7.$
6. $m^2 + 6m = 16.$
7. $x^2 + 7x - 44 = 0.$
8. $60 = 4y + y^2.$

In examples 9–12 leave answers in radical form.

9. $2y^2 - 12y = 15.$
10. $2n^2 - 10n = 1.$
11. $3x^2 = x + 3.$
12. $4x^2 - 2 = 3x.$

Group B

Solve each of the following correct to the nearest tenth.

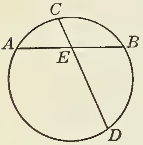
13. $x^2 + 3x = 1.$
14. $x^2 = x + 3.$
15. $x^2 - x - 1 = 0.$
16. $m^2 - 4m = 6.$
17. $x^2 - 3x + \frac{3}{4} = 0.$
18. $x^2 - 2x = -\frac{1}{2}.$
19. $x^2 - \frac{4}{3}x = 3.$
20. $2m^2 - 5m + 1 = 0.$
21. $5x^2 + 2x = 2.$
22. $4y^2 = 3y + 4.$
23. $3m^2 - 6m = 5.$
24. $6x^2 = 5 - 3x.$

Group C

25. The distance traveled by an object thrown vertically upward is given by the formula $D = 64t - 16t^2$. After how many seconds (to the nearest tenth) will the object be 13 feet above the ground? Discuss the two answers.

26. In geometry you learned that if from a point P outside a circle, tangent PA and a secant PB cutting the circle at C be drawn then $\overline{PA}^2 = PB \times PC$. If $PA = 10$ and $BC = 6$, find PB to the nearest tenth.

27. Geometry teaches us that if two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other. In the adjoining figure, then, $AE \times EB = CE \times ED$. If $AE = 5$, $EB = 6$, and $CD = 13$, what is the value of ED?



28. One leg of a right triangle is 3 feet more than the other. If the hypotenuse is 10 feet, find the length of each leg correct to the nearest tenth.

29. If, in a right triangle ABC, the altitude CD to the hypotenuse is drawn, then $\overline{AC}^2 = AB \times AD$. If AC is 15 and DB exceeds AD by 5, find the value of AD correct to the nearest tenth.

A general solution, the formula. As mentioned previously, the general or standard form of a quadratic equation is $ax^2 + bx + c = 0$, since every quadratic equation may be written in this form. In the general form of the quadratic equation the letter a always stands for the coefficient of x^2, the letter b for the coefficient of x, and c for the constant term.

Arrange the following equations in the general form and state the values of a, b, and c in each equation:

(a) $2x^2 + 5x = 7$.	(d) $x^2 + 3x = 4$.
(b) $3x^2 + 7 = 2x$.	(e) $x^2 + x = 3$.
(c) $4x - 15 = 3x^2$.	(f) $3x^2 = x + 1$.

Now it would certainly save labor if we could discover a solution of this general equation which could be used as a formula for finding the roots of any quadratic equation.

Let us solve the general equation by the method of completing the square.

$$ax^2 + bx + c = 0.$$

Dividing by a $\qquad x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$

Transposing $\frac{c}{a}$ $\qquad x^2 + \frac{b}{a}x = -\frac{c}{a}.$

Now to complete the square, add the square of one-half the coefficient of x to each side.

$$\left[\frac{1}{2} \text{ of } \frac{b}{a} = \frac{b}{2a};\ \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}\right].$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}.$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking the square root of each member,

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

As expected, you will observe that we have two roots, caused by the separate use of the $\pm$ before the radical. From now on when you cannot solve a quadratic equation quickly by factoring, turn immediately to the method of the formula.

Remember

The roots of the equation $ax^2 + bx + c = 0$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Illustrative example. Find the roots of $3x^2 - 5x = 7$ correct to the nearest tenth.

Solution

Write the equation in the form $ax^2 + bx + c = 0$ in which a is positive.

$$3x^2 - 5x - 7 = 0.$$

Comparing this equation with the general equation $ax^2 + bx + c = 0$, we find that $a = 3$, $b = -5$, and $c = -7$. Substituting these values in

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-7)}}{2(3)}$$

$$= \frac{5 \pm \sqrt{25 + 84}}{6}\,*$$

$$= \frac{5 \pm \sqrt{109}}{6}.$$

$$\therefore x = \frac{5 \pm 10.44}{6}.$$

$x = \frac{5 + 10.44}{6} = \frac{15.44}{6}$	$x = \frac{5 - 10.44}{6} = \frac{-5.44}{6}$
$= 2.57$	$= -.90$
$= 2.6.$	$= -.9.$

$$x = 2.6, -.9.$$

Check

$3(2.6)^2 - 5(2.6) = 7.$	$3(-.9)^2 - 5(-.9) = 7.$
$20.28 - 13 = 7.$	$2.43 + 4.5 = 7.$
$7.28 = 7$ [approximately].	$6.93 = 7$ [approximately].

* At this point you can test the accuracy of your signs as follows:

1. The sign of the x term is the opposite of the sign of the first number of the answer.
2. The sign of the constant term is the opposite of the last sign under the radical.

$$3x^2 - 5x \qquad - 7 = 0.$$

$$x = \frac{+5 \pm \sqrt{25 + 84}}{6}.$$

EXERCISES

Group A

Solve each of the following quadratic equations by the formula:

1. $x^2 - 3x + 2 = 0.$
2. $x^2 - 6x + 9 = 0.$
3. $x^2 + 3x + 2 = 0.$
4. $x^2 + x = 2.$
5. $x^2 - x = 12.$
6. $3x^2 + 8x = 3.$
7. $2y^2 = 3y + 20.$
8. $7m^2 - 2 = -5m.$
9. $x^2 + 1.5 = 2.5x.$
10. $x^2 + 1.5x = 1.$
11. $x^2 + .2x = 1.68.$
12. $x^2 + .5x - .84 = 0.$

Group B

Solve examples 13 to 17 correct to the nearest tenth, and 18 to 22 correct to the nearest hundredth:

13. $x^2 - x = 1.$
14. $y^2 - 4y = 6.$
15. $m^2 - 3 = m.$
16. $x^2 + 5x = 3.$
17. $2x^2 - 10x = 9.$
18. $2x^2 + 1 = 4x.$
19. $4y^2 = 8y + 3.$
20. $3x^2 + 7x = 9.$
21. $3x^2 - 10x = 3.$
22. $6m^2 + 10m = 7.$

Group C

When possible, solve these equations by factoring; otherwise use the formula, computing the roots correct to the nearest tenth:

23. $3x = 2 + \frac{2}{x}.$
24. $3x^2 - \frac{x}{4} = \frac{7}{2}.$
25. $\frac{x-1}{x} + \frac{x}{x-1} = \frac{5}{2}.$
26. $\frac{x^2}{x-1} = \frac{1}{1-x} + 10.$
27. $\frac{x}{x+8} = \frac{x+3}{2x+1}.$
28. $\frac{2x+3}{x-1} - 6 = \frac{5}{x^2+2x-3}.$

Imaginary numbers. Let us consider the solution of

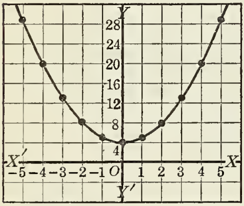
$$x^2 + 4 = 0.$$

Transposing, $x^2 = -4$ and $x = \pm\sqrt{-4}$. What does $\sqrt{-4}$ mean? What is its value? Is it equal to $+2$? Evidently not, because $(+2)^2 = +4$. Is it equal to -2? No, because $(-2)^2 = +4$. So you see the equation $x^2 + 4 = 0$ cannot be satisfied by either $+2$ or -2.

Let us draw the graph of $x^2 + 4 = 0$ and see if it throws any light on this new number. Putting $y = x^2 + 4$, and making a table, we have

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	29	20	13	8	5	4	5	8	13	20	29

The graph below represents the equation $y = x^2 + 4$. In our previous work we learned that the roots of the equation $x^2 + 4 = 0$ are the abscissas of the points where the parabola crosses the x-axis. But this graph does not cut the x-axis. This confirms our suspicion that a new number is introduced by this equation.



Before proceeding any further, let us briefly recall the kinds of numbers with which we have worked. In arithmetic you used whole numbers, *integers.* Then by dividing one integer by another you became acquainted with *fractions.* Then in the beginning of your study of algebra you worked with *negative numbers.* Those numbers which are either positive or negative integers, or fractions whose terms are such integers, are called *rational numbers.*

Later on in finding square roots we met a new kind of number, $\sqrt{2}$ and $\sqrt{6}$. These numbers are not rational because we cannot extract their square roots exactly. Such numbers are called *irrational numbers.* An irrational number cannot be written as a fraction of two integers. Repeating decimals, however, as .3333 · · ·, you will learn later, can be expressed as fractions and are therefore rational. Thus the fact that there is no end to the decimal is no proof that the number is irrational. Both rational and irrational numbers are called *real numbers.*

Now returning to our discussion we see that $\pm\sqrt{-4}$ is neither a rational nor an irrational number. It is a new kind of number called an *imaginary number* and is defined as the *indicated even root of a negative number.*

You should not be misled by the word *imaginary.* For many years mathematicians were baffled by this new number and for want of a better word called it an imaginary number. However, notwithstanding the advances made in understanding this number, the name, unfortunately, is still used. Imaginary numbers have served a very useful purpose just as the other kinds of number have, and are no less useful than the numbers we call *real.* In fact, it would have been very difficult to have made certain discoveries in electricity without the imaginary number.

A further discussion of the imaginary number will be taken up in Chapter ~~XI.~~ p. 328.

General properties of the quadratic equation. In order to help you in your later work in mathematics, we shall now study the relation which exists between the coefficients and roots of a quadratic equation.

It is often desirable to determine, without actually solving, whether the roots of a given quadratic equation are real or imaginary, rational or irrational, equal or unequal. The "character" of the roots of a quadratic equation can be determined by a study of the formula used for solving quadratic equations.

The roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In other words, if r_1 and r_2 represent the roots, then

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Now it can easily be seen that the nature of the roots will be affected if the quantity under the radical sign is positive or negative since this is the only part of the formula that tells us whether the roots are rational, irrational, or imaginary. In fact, a careful examination of the value of $b^2 - 4ac$ determines the character of the roots of the given equation. This expression, $b^2 - 4ac$, which decides the character or nature of the roots of a quadratic equation is called the *discriminant* and is frequently represented by the Greek letter Δ (delta). (It is assumed throughout the following discussion that the coefficients of the quadratic equations considered are real and rational numbers.)

1. *When are the roots real?* When the value of $b^2 - 4ac$ is a positive number or zero, the roots will be real.

Thus
$$r_1 = \frac{-b + \sqrt{\text{pos. number or zero}}}{2a};$$
$$r_2 = \frac{-b - \sqrt{\text{pos. number or zero}}}{2a}.$$

2. *When are the roots imaginary?* We have seen that the square root of a negative number produces an imaginary number. Therefore, if $b^2 - 4ac$ is negative, the roots are imaginary numbers.

Thus $r_1 = \dfrac{-b + \sqrt{\text{neg. number}}}{2a}$; $r_2 = \dfrac{-b - \sqrt{\text{neg. number}}}{2a}$.

3. *When are the roots rational?* If $b^2 - 4ac$ is a perfect square or zero, the radical sign vanishes because its exact square root can be extracted.

4. *When are the roots irrational?* If $b^2 - 4ac$ is not a perfect square, its square root cannot be exactly extracted and so the roots are irrational.

5. *When are the roots equal?* If $b^2 - 4ac$ is zero, then we shall have $r_1 = \dfrac{-b+\sqrt{0}}{2a}$ and $r_2 = \dfrac{-b-\sqrt{0}}{2a}$.

Hence r_1 and r_2 are each equal to $\dfrac{-b}{2a}$ and are therefore equal.

6. *When are the roots unequal?* If $b^2 - 4ac$ is not equal to zero, the radical does not disappear and r_1 and r_2 will have different values making them unequal.

The above conclusions may be summarized as follows:

If $b^2 - 4ac$ is	the roots of $ax^2 + bx + c = 0$ are	In all other cases they are
negative,	imaginary.	real.
zero,	equal.	unequal.
a square,	rational.	irrational.

Illustrative examples.

Example 1. Find the character of the roots of

$$3x^2 - 10x + 3 = 0.$$

Solution

In this equation $a = 3$, $b = -10$, $c = 3$.

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= 100 - 36 = +64.\end{aligned}$$

$\therefore$ The roots are:

1. Real (Δ is positive).
2. Rational (Δ is a perfect square).
3. Unequal ($\Delta \neq 0$).

Example 2. Find the character of the roots of the equation

$$2x^2 - 3x - 1 = 0.$$

Solution

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= 9 + 8 = +17. \end{aligned}$$

$\therefore$ The roots are:

1. Real (Δ is positive).
2. Irrational (Δ not a perfect square).
3. Unequal ($\Delta \neq 0$).

Example 3. Find the character of the roots of the equation

$$x^2 + 6x + 9 = 0.$$

Solution

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= 36 - 36 = 0. \end{aligned}$$

$\therefore$ The roots are:

1. Real 2. Rational 3. Equal

Example 4. Find the character of the roots of

$$x^2 + 2x + 7 = 0.$$

Solution

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= 4 - 28 = -24. \end{aligned}$$

$\therefore$ The roots are imaginary. (Δ is negative.)

EXERCISES

Group A

Without solving, determine the character of the roots of:

1. $4x^2 - 4x + 1 = 0.$

2. $y^2 - 3y - 4 = 0.$

3. $x^2 + x + 2 = 0.$

4. $3y^2 + 4y - 10 = 0.$

5. $4x^2 + 9 = 12x.$

6. $x^2 - 5x - 75 = 0.$

7. $3x^2 + 8x - 3 = 0.$

8. $2y^2 + 3y = 0.$

9. $2x^2 + 6x + 3 = 0.$

10. $7m^2 + 15m + 2 = 0.$

11. $9x - 7 - 3x^2 = 0.$

12. $25x^2 - 3092x - 14400 = 0.$

Draw the graph of each of the following and from it determine the nature of the roots. Then compute the discriminant and check:

13. $x^2 - 4x + 3 = 0.$

14. $x^2 - 5x = 0.$

15. $4x^2 - 12x + 9 = 0.$

16. $3x^2 + 2x + 1 = 0.$

17. $4x^2 = 9.$

18. $2x^2 + 5 = 0.$

Group B

From the discriminant tell the character of the roots and whether the graph crosses, touches, or neither crosses nor touches the x-axis:

19. $4x = 9 - x^2.$

20. $4x = 12 - x^2.$

21. $-4x = 12 + x^2.$

22. $-4x = 1 + 4x^2.$

23. $x(x - 5) = x - 16.$

24. $4x^2 + 20x = -25.$

Illustrative example. Determine the values of k which will make the roots of $x^2 + kx + 36 = 0$ equal.

Solution

Since the roots are to be equal, the discriminant must equal zero.

Hence
$$\begin{aligned} b^2 - 4ac &= 0. \\ k^2 - 144 &= 0. \\ k^2 &= 144. \\ k &= \pm 12. \end{aligned}$$

Check

Substituting $+12$ for k, $\quad x^2 + 12x + 36 = 0.$

The roots of this equation are -6 and -6, *i.e.*, equal.

In the same way check $k = -12$.

Determine the value or values of k which will make the roots of the following equations equal.

25. $x^2 + kx + 49 = 0.$

26. $kx^2 - 2x + 1 = 0.$

27. $x^2 - 12x + k = 0.$

28. $x^2 - kx - (k + 8) = 0.$

29. $x^2 - kx + k + 3 = 0.$

30. $(k - 2)x^2 - kx = -2.$

Group C

31. Find the value of n so that 2 shall be a root of $x^2 - nx + 2 = 0$.

32. If one root of $x^2 + kx - 24 = 0$ is -2, determine k.

33. If the roots of an equation are real and unequal, b^2 must be ? than $4ac$; if the roots are equal, b^2 must be ? to $4ac$; if the roots are imaginary, b^2 must be ? than $4ac$.

34. For what values of k will the roots of $kx^2 + 4x + 1 = 0$ be (a) real and unequal, (b) real and equal, (c) imaginary?

35. For what values of k will the roots of $x^2 + 6x + k = 0$ be (a) real and unequal, (b) real and equal, (c) imaginary?

36. For what values of k will the roots of $3x^2 - 4x - k = 0$ be imaginary?

Relations between roots and coefficients. We have seen that the roots of the general quadratic equation $ax^2 + bx + c = 0$ may be written

$$(1)\ r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } (2)\ r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

A further study of these roots reveals two more important facts.

If we add (1) and (2), we have

$$\begin{aligned} r_1 + r_2 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a} \\ &= -\frac{2b}{2a} = -\frac{b}{a}. \end{aligned}$$

If we multiply (1) and (2), we have

$$\begin{aligned} r_1 r_2 &= \frac{(-b + \sqrt{b^2 - 4ac})(-b - \sqrt{b^2 - 4ac})}{(2a)\qquad\qquad(2a)} \\ &= \frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}. \end{aligned}$$

In a quadratic equation: (1) *The sum of the roots* $= -\frac{b}{a}$, *and* (2) *the product of the roots* $= \frac{c}{a}$.

Illustrative example. Determine by inspection, the sum and product of the roots of $2x^2 - 7x + 3 = 0$.

Solution

The sum of the roots $= -\frac{b}{a} = -\frac{-7}{2} = +3\frac{1}{2}$.

The product of the roots $= \frac{c}{a} = \frac{3}{2} = +1\frac{1}{2}$.

EXERCISES

Determine by inspection, the sum and the product of the roots of each of the following equations:

1. $x^2 + 3x - 21 = 0$.
2. $y^2 + 6y + 9 = 0$.
3. $x^2 - 5x = 6$.
4. $2m^2 - 5m + 2 = 0$.
5. $3x^2 + 3 = 10x$.
6. $10x + 5 = -3x^2$.
7. $2y^2 - 5y + 7 = 0$.
8. $6x^2 + 11x + 3 = 0$.

Formation of a quadratic equation from its roots. We have just learned that $r_1 + r_2 = -\frac{b}{a}$ and $r_1r_2 = \frac{c}{a}$. If we divide both sides of the general quadratic equation $ax^2 + bx + c = 0$ by a, the coefficient of x^2, we have

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \tag{1}$$

Now if $(r_1 + r_2) = -\frac{b}{a}$, then $\frac{b}{a} = -(r_1 + r_2)$.

Substituting $-(r_1 + r_2)$ for $\frac{b}{a}$ and r_1r_2 for $\frac{c}{a}$ in equation (1), we obtain the general quadratic equation

$$x^2 - (r_1 + r_2)x + r_1r_2 = 0 \tag{2}$$

in terms of its roots.

Again if the parenthesis in equation (2) is removed, the terms rearranged and factored, we get

$$(x - r_1)(x - r_2) = 0 \qquad (3)$$

another form of the quadratic equation in terms of its roots. Both (2) and (3) may be used to form equations when given the roots, although (2) is better suited for problems involving radicals.

Illustrative examples.

Example 1. Form the equation whose roots are 2 and -3.

Solution

In this equation $r_1 = 2$ and $r_2 = -3$.

Method I. $\quad x^2 - (r_1 + r_2)x + r_1r_2 = 0.$

Substituting $\quad x^2 - (2 - 3)x + (2)(-3) = 0.$

$$x^2 + x - 6 = 0.$$

Method II. $\quad (x - r_1)(x - r_2) = 0.$

Substituting $\quad [x - (2)][x - (-3)] = 0.$

$$(x - 2)(x + 3) = 0.$$

$$x^2 + x - 6 = 0.$$

Example 2. Form the equation whose roots are $3 + \sqrt{2}$ and $3 - \sqrt{2}$.

Solution

$$x^2 - (r_1 + r_2)x + r_1r_2 = 0.$$

Substituting

$$x^2 - (3 + \sqrt{2} + 3 - \sqrt{2})\,x + (3 + \sqrt{2})(3 - \sqrt{2}) = 0.$$

$$x^2 - 6\,x + 7 = 0.$$

EXERCISES

Form the equations whose roots are:

1. 3, 4.
2. -2, 1.
3. 7, -2.
4. -2, -3.
5. $-a$, $2a$.
6. b, $-b$.

7. $\frac{1}{2}, \frac{2}{3}$.

8. $-\frac{3}{4}, \frac{7}{8}$.

9. .3, .5.

10. $-.2, 1.5$.

11. $\sqrt{2}, -\sqrt{2}$.

12. $1+\sqrt{2}, 1-\sqrt{2}$.

13. $\sqrt{a}, -\sqrt{a}$.

14. $b+\sqrt{a}, b-\sqrt{a}$.

15. $\sqrt{3}+\sqrt{2}, \sqrt{3}-\sqrt{2}$.

16. $\frac{\sqrt{5}+\sqrt{2}}{3}, \frac{\sqrt{5}-\sqrt{2}}{3}$.

17. $\frac{\sqrt{3}-\sqrt{2}}{2}, \frac{\sqrt{3}+\sqrt{2}}{2}$.

18. $\frac{\sqrt{5}-\sqrt{3}}{5}, \frac{\sqrt{5}+\sqrt{3}}{5}$.

19. $\frac{5+\sqrt{6}}{2}, \frac{5-\sqrt{6}}{2}$.

20. $\frac{2+\sqrt{1.2}}{3}, \frac{2-\sqrt{1.2}}{3}$.

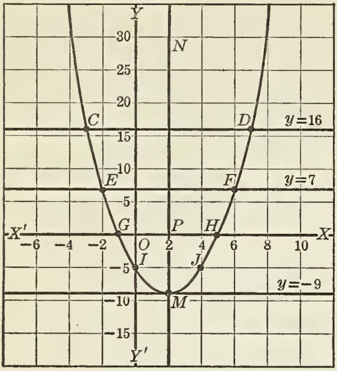
Maxima and Minima

Problem. Parcel post regulations state that the combined length and girth (perimeter of one end) of a certain type of package cannot exceed 72 inches. If the package is to be a box 18 inches wide, what is the length and height of the largest box that can be sent?

This problem, involving the dimensions of the largest box under certain conditions, is one of a large number of similar problems vitally important to everyone. We find the same idea of " most " and " least " in everyday life. Your mother and father when shopping naturally are anxious to get the " most " for their money, or to shop where they pay " least " for an article or commodity. Business men find that the success of an enterprise depends largely upon the " maximum " output and the " minimum " cost and overhead. We all are, in some way or other, deeply concerned with problems of " most " and " least " values or, as termed in mathematics, *maximum* and *minimum* values.

A study of the graphic and algebraic solutions of the quadratic equation will help us considerably in understanding the method of solving maximum and minimum problems.

Graphs and Maxima and Minima. The graph of the quadratic equation $x^2 - 4x - 5 = y$ is illustrated in the adjoining diagram. The increase in x from -5 to $+9$ is noted by moving along the x-axis. Now let us see what changes take place in y as x increases.



1. As x increases from -5 to -1 the curve descends and the value of y ? (increases or decreases).

2. As x increases from -1 the curve descends until it reaches the lowest point labelled ? .

3. The abscissa of point M is ? and the ordinate of point M is ? .

4. As x increases from $+2$, the curve turns up (ascends) and the value of y ? (increases or decreases).

The point M in the diagram is the lowest point of the curve. This point is called the "turning point" and gives us the minimum or smallest value of y (ordinate PM). Observe that MN, the axis of symmetry, passes through the "turning point."

Now from the graph of the equation $x^2 - 4x - 5 = y$ we have learned that we may find the roots of the equation $x^2 - 4x - 5 = 16$ by noting the abscissas of the points where the parabola crosses the line $y = 16$ (see points C and D). Similarly for the roots of the equations $x^2 - 4x - 5 = 7$ (points E and F); $x^2 - 4x - 5 = 0$ (points G and H); $x^2 - 4x - 5 = -5$ (points I and J); and $x^2 - 4x - 5 = -9$ (point M).

Summarizing we have the table on the opposite page.

Equation	Roots	Sum of Roots
$x^2 - 4x - 5 = 16$	-3 and 7	$+4$
$x^2 - 4x - 5 = 7$	-2 and 6	$+4$
$x^2 - 4x - 5 = 0$	-1 and 5	$+4$
$x^2 - 4x - 5 = -5$	0 and 4	$+4$
$x^2 - 4x - 5 = -9$	2	$+4$

Observe that in each case, the sum of the roots is $+4$, twice the value of x at the lowest point. That is, the abscissa of the minimum point is half the sum of the roots, $\left(\text{half of } -\frac{b}{a}\right)$, or $-\frac{b}{2a}$. It is also evident from the figure that the abscissa of the lowest point is the average (half the sum) of the roots of any of these equations.

$$\therefore \text{ For the minimum point, } x = -\frac{b}{2a}.$$

The ordinate of the turning point can be found by substituting this value of x in the equation $y = ax^2 + bx + c$.

Something to think about.

1. Does the axis of symmetry of a parabola always pass through the turning point?

2. Since the abscissa of the turning point is $-\frac{b}{2a}$, what is the equation of the axis of symmetry?

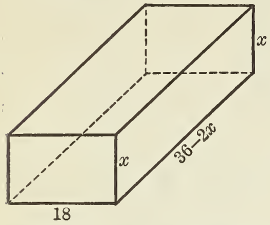
Now let us return to the problem given at the beginning of this discussion as an

Illustrative example. Parcel post regulations state that the combined length and girth (perimeter of one end) of a certain type of package cannot exceed 72 inches. If the package is to be a box 18 inches wide, what is the length and height of the largest box that can be sent?

Solution

Let x = the height in inches.

Then since the width is 18″ and the combined length and girth is 72 inches, the length must equal $72 - (36 + 2x) = 36 - 2x$.



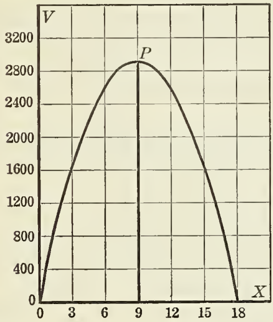
Then volume $V = 18x(36 - 2x)$.

$= 648x - 36x^2$.

∴ We wish to know the maximum value of $648x - 36x^2$.

1. *By means of a graph.*

Giving values to x (x cannot be negative) and computing the corresponding values of V, we obtain the table and the graph below:



If x =	Then V =
0	0
3	1620
6	2592
9	2916
12	2592
15	1620
18	0

The maximum point P has its abscissa = 9.

∴ x or height = 9″ and length = $36 - 2x$ = 18″.

2. *By means of algebra.*

In the equation $V = 648x - 36x^2$, $a = -36$ and $b = 648$.

The abscissa of the turning point $= -\dfrac{b}{2a} = -\dfrac{648}{-72} = 9$.

∴ Dimensions required are height = 9″, and length = $(36 - 2x)$ = 18″.

EXERCISES

Group A

Determine the maximum or minimum value of each of the following quadratic equations by the graphic method and check by the algebraic method:

1. Given the equation $y = 9 + 8x - x^2$:

(*a*) Draw the graph of the function from $x = -3$ to $x = 9$.

(*b*) Indicate by point P the position of a maximum or minimum point.

(*c*) State the coördinates (abscissa and ordinate) of this turning point by inspection, and check by the algebraic formulas.

(*d*) Draw the axis of symmetry, and write its equation.

2. Given the equation $y = x^2 - 2x - 4$:

(*a*) Draw the graph of this function.

(*b*) Indicate on the graph by point P the turning point of the graph.

(*c*) Is this turning point a maximum or minimum value?

(*d*) State the coördinates of this turning point by inspection and check by the algebraic formulas.

(*e*) Draw the axis of symmetry, and write its equation.

3. $y = x^2 - 6x + 5$.

4. $y = x^2 + 6x - 5$.

5. $y = 3x - x^2$.

6. $y = x^2 + 10x + 25$.

7. $y = 2x^2 - x - 3$.

8. $y = 8 + 2x - x^2$.

9. $y = 2x^2 + 7x + 2$.

10. $y = -x^2 - 2x + 8$.

11. $y = x^2 - 5$.

12. $y = -2x^2 + 8x - 3$.

Group B

Determine without drawing the graph of the following equations: (*a*) the coördinates of the maximum or minimum point; (*b*) the equation of the axis of symmetry.

13. $y = x^2 - 12x + 72$.

14. $d = 96t - 16t^2$.

15. $P = 60x - 2x^2$.

16. A gardener has 200 feet of wire fencing. What are the dimensions of a rectangular garden which can be enclosed with this amount of fencing, so as to give the largest area?

17. If a baseball is thrown upward with a velocity of 112 feet per second, the height reached after t seconds is given by the formula $h = 112\,t - 16\,t^2$. Find the greatest height which the ball reaches.

18. Divide 20 into two parts so that the sum of the squares of these two parts will have the least possible value.

19. A farmer has 40 yards of fencing and wishes to enclose a rectangular garden, using the side of a barn as one side. What are the dimensions of the garden of largest area he can enclose?

20. Divide a line 20 inches long into two parts such that if these two parts be taken as the arms of a right triangle, the hypotenuse will be the least possible length.

Group C

21. A rectangular field is 12 rods long and 8 rods wide. If the length is to be decreased and the width increased by the same amount, what must this amount be if the new area shall be the largest possible?

22. A sheet of tin 12″ by 8″ is to be made into an open box by cutting out equal squares from the corners and folding up the flaps. Find the length of the side of each of the small squares cut out in order that the sum of the four sides of the box may have the largest possible area.

23. Divide a line $2\,a$ inches long into two parts such that the rectangle having these two parts as dimensions shall have the greatest possible area. What is the name given to this type of rectangle?

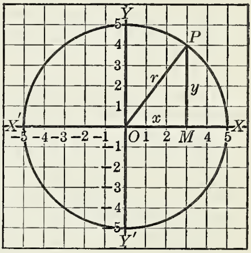
24. The area of a triangle in terms of two sides and the included angle is given by the formula $K = \frac{1}{2}\,bc \sin A$. If two sides of a triangle are 12 and 6, what is the value of the included angle so that the triangle shall have the largest possible area? (Determine the number of degrees by inspection.)

25. Find one angle of a parallelogram, having two adjacent sides 18 and 10 so that the area shall be the largest possible ($K = bc \sin A$).

The circle. To study the properties of a circle let us draw the graph of $x^2 + y^2 = 25$. First we must solve for y in terms of x, thus, $y = \pm\sqrt{25 - x^2}$. Giving x values and computing the corresponding values of y, we have the following table:

If $x =$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
Then $y =$	imag.	0	± 3	± 4	± 4.6	± 4.9	± 5	± 4.9	± 4.6	± 4	± 3	0	imag.

Something to think about. Is y real or imaginary for values of x greater than 5? Why? Does the curve then extend to the right of 5? How far to the left does it extend?



Plotting the points, and using the same scale on the x-axis as on the y-axis, we have the accompanying graph. What is the shape of this curve? Where is its center? What is its radius? Could the radius be obtained from the constant term? How?

The graph of $x^2 + y^2 = r^2$ is a circle whose radius is r and whose center is at the origin.

To prove: The equation of a circle (radius r, center O) is $x^2 + y^2 = r^2$.

Take any point P on the curve and let $OM = x$ and $PM = y$. Then, since $\overline{OM}^2 + \overline{PM}^2 = \overline{OP}^2$, $x^2 + y^2 = r^2$.

EXERCISES

From each of the following equations (*a*) determine the radius of the circle and (*b*) draw the graph:

Group A

1. $x^2 + y^2 = 4.$
2. $x^2 + y^2 = 9.$
3. $x^2 + y^2 = 16.$
4. $x^2 + y^2 = 25.$
5. $x^2 + y^2 = 36.$
6. $x^2 + y^2 = 49.$
7. $x^2 + y^2 = 64.$
8. $x^2 + y^2 = 121.$

Group B

9. $x^2 + y^2 = \frac{25}{16}.$
10. $x^2 + y^2 = 6.25.$
11. $x^2 + y^2 = 2\frac{1}{4}.$
12. $2x^2 + 2y^2 = 98.$
13. $2x^2 + 2y^2 = 18.$
14. $8x^2 + 8y^2 = 18.$

Group C

15. $x^2 + y^2 = 12.$
16. $x^2 + y^2 = 20.$
17. $y^2 = 18 - x^2.$
18. $x^2 = 32 - y^2.$
19. $3x^2 + 3y^2 = 18.$
20. $5x^2 + 5y^2 = 36.$

The ellipse. The ellipse is a curve of great importance and interest. It is very common in nature and has extensive use in the arts and sciences. The earth's orbit about the sun and the orbits of all the other planets are ellipses. The ellipse is found in bridge construction (elliptical arches) and in the construction of auditoriums and "whispering galleries." Elliptical gears are found in machine construction because of certain properties of the curve. In everyday life we frequently refer to "ovals" in the form of ellipses in design work.

To study the properties of an ellipse let us graph the equation

$$9x^2 + 16y^2 = 144.$$

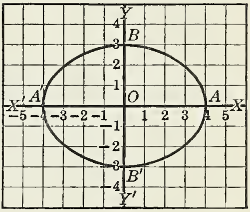
First, solving for y in terms of x, we have

$$y = \pm\sqrt{\frac{144 - 9x^2}{16}} \quad \text{or} \quad y = \pm \tfrac{1}{4}\sqrt{144 - 9x^2}.$$

Now assigning values to x and finding the corresponding values of y, we have the table:

If $x =$	-5	-4	-3	-2	-1	0	1	2	3	4	5
Then $y =$	imag.	0	± 1.9	± 2.6	± 2.9	± 3	± 2.9	± 2.6	± 1.9	0	imag.

Graphing these points, we obtain the accompanying diagram.



Observations from the graph.

1. An ellipse has two axes of symmetry perpendicular to each other. $A'A$ is called the *major axis* and BB' the *minor axis*.

2. Values of x greater than 4 would make y imaginary.

3. By letting $x = 0$, we obtain the y-intercepts, ± 3, and by letting $y = 0$, we obtain the x-intercepts, ± 4.

Something to think about. How would you determine the lengths of the major axis and the minor axis from the equation of the ellipse?

If the center of the ellipse is the origin, the equation has the form $ax^2 + by^2 = c$ where a, b, and c are of the same sign and not equal to zero, but a and b are different in absolute value.

EXERCISES

Draw the graphs of each of the following equations.

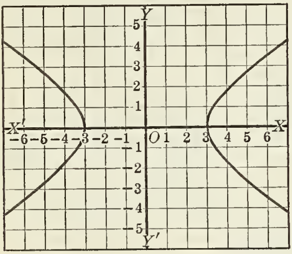
1. $x^2 + 4y^2 = 16$.
2. $9x^2 + y^2 = 4$.
3. $4x^2 + 9y^2 = 16$.
4. $25x^2 + 4y^2 = 100$.
5. $x^2 + 3y^2 = 3$.
6. $3x^2 + 4y^2 = 12$.

The hyperbola. To study the properties of an hyperbola let us graph $4x^2 - 9y^2 = 36$. Solving for y in terms of x, we have

$$y = \pm\sqrt{\frac{4x^2 - 36}{9}} = \pm\sqrt{\frac{4(x^2 - 9)}{9}} = \pm\tfrac{2}{3}\sqrt{x^2 - 9}.$$

Assigning values to x and computing the corresponding values of y gives the table on the right from which we draw the graph below.

If $x =$	Then $y =$
-6	± 3.5
-5	± 2.7
-4	± 1.7
-3	0
-2	imag.
-1	imag.
0	imag.
1	imag.
2	imag.
3	0
4	± 1.7
5	± 2.7
6	± 3.5



Observations from the graph.

1. Values of x between $+3$ and -3 make y imaginary, showing that the curve does not exist between these points.
2. The curve has two branches.
3. The graph is symmetrical with respect to both axes.

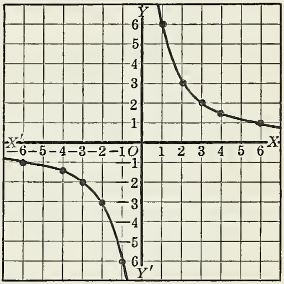
The curve is of the form $ax^2 - by^2 = c$ and is called an hyperbola. The coefficients of x^2 and y^2 are not zero, are opposite in sign, and may or may not be equal in absolute value; c may have any positive or negative value, but not 0.

Let us also graph the equation $xy = 6$. Solving for y in terms of x, we have $y = \frac{6}{x}$. From this equation, we have the table and graph at the top of the opposite page.

If $x =$	0	1	2	3	4	6	-1	-2	-3	-4	-6	∞ *
Then $y =$	∞	6	3	2	1.5	1	-6	-3	-2	-1.5	-1	0

Observations from the graph.

1. As the values of x become smaller, in absolute value, getting nearer and nearer to zero, the values of y increase in absolute value and the curve approaches the y-axis. Similarly as the values of y become smaller in absolute value the values of x increase in absolute value and the curve approaches the x-axis.



2. The curve is symmetrical with respect to the origin.

3. The curve has two branches, one in the first quadrant and one in the third quadrant. (If the constant term is negative, the branches will lie in the second and fourth quadrants.)

This curve is also an hyperbola and is of the form $xy = k$ where k has any positive or negative value except zero.

The graphs of hyperbolas are frequently used in scientific work and in making business calculations.

EXERCISES

Draw the graph of each of the following:

1. $x^2 - 4y^2 = 16$.
2. $9x^2 - y^2 = 4$.
3. $4x^2 - 9y^2 = 16$.
4. $x^2 - y^2 = 4$.
5. $xy = 12$.
6. $xy = -8$.

* From the equation $xy = 6$, we derive $x = \frac{6}{y}$. Letting $y = 0$, we obtain $x = \infty$.

REVIEW OF CONIC SECTIONS

1. Name the graph of each of the following:

(a) $xy = 7.$
(b) $x^2 - y^2 = 20.$
(c) $2x^2 - y^2 = 14.$
(d) $x = y^2 + 1.$
(e) $9x^2 + 4y^2 = 36.$
(f) $y^2 = x.$
(g) $x^2 + y^2 = 40.$
(h) $x = \frac{6}{y}.$
(i) $y = x^2 - 9.$
(j) $x^2 + y^2 = 10.$
(k) $2x^2 + 2y^2 = 36.$
(l) $xy = -5.$
(m) $x^2 = y^2 + 10.$

2. Write the equation whose graph is a circle of radius 5.

3. How can you tell from its equation whether a graph will be symmetrical to the y-axis? To the x-axis?

4. Write an equation whose graph will be an hyperbola with branches in the first and third quadrants; in the second and fourth quadrants.

5. In the distance formula, $d = rt$, suppose the distance equals 50. Then the equation becomes $50 = rt$. The graph of this equation will be ? .

6. How does an increase in x in the equation $y = \frac{K}{x}$ affect y?

7. Write the equation of a circle whose center is the origin and whose radius is 7; $\sqrt{7}$.

8. What are the general characteristics of the graph of the function $y = ax^2 + bx + c$, if a is negative?

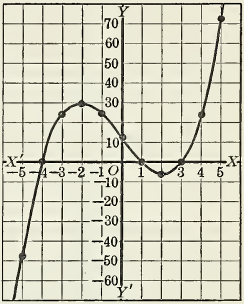
9. For what real value of k does the graph of the equation $x^2 + 6x - k + 3 = 0$ pass through the origin?

Graphs of Equations of Higher Degree

The method of graphing second-degree equations can be extended to equations of higher degree than the second. For example, graph the equation $y = x^3 - 13x + 12$.

Assigning values to x and computing the corresponding values of y, we have the table and graph below:

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	-126	-48	0	24	30	24	12	0	-6	0	24	72



Questions on the graph.

1. In how many points does the curve cut the x-axis? What are the abscissas of these points?

2. Then what are the roots of the equation

$$x^3 - 13x + 12 = 0?$$

3. What are the coördinates of the maximum and minimum points on the curve?

EXERCISES

Find the roots of each of the following equations from its graph and check each root.

1. $x^3 - 3x^2 - x + 3 = 0.$
2. $x^3 - 4x = 0.$
3. $x^3 + 3x^2 - x - 3 = 0.$
4. $x^3 - x^2 - 12 = 0.$
5. $x^4 - 20x^2 + 64 = 0.$
6. $x^4 - 2x^3 - 11x^2 + 12x = 0.$

Graphic Representation of Trigonometric Functions

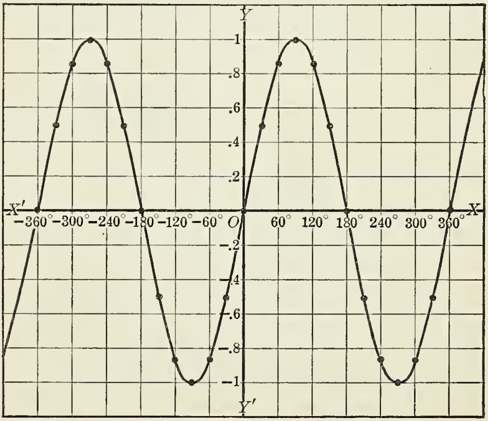
In graphing trigonometric functions we proceed in a manner similar to that learned in this chapter. We assume values for the angle and these angles or the radian measures of these angles are taken as the abscissas. The corresponding values of the function of these angles are found from the table on page 3, and these are taken as the ordinates. If the angle is negative or is greater than 90°, we use the reduction formulas learned in the last chapter.

The graph of $y = \sin x$. Letting x have values and computing the corresponding values of y, we have the following tables:

If $x =$	or	Then $y =$
0°	0	0
30°	$\frac{\pi}{6}$	.5
60°	$\frac{\pi}{3}$	.86
90°	$\frac{\pi}{2}$	1
120°	$\frac{2\pi}{3}$	.86
150°	$\frac{5\pi}{6}$	.5
180°	π	0
210°	$\frac{7\pi}{6}$	− .5
240°	$\frac{4\pi}{3}$	− .86
270°	$\frac{3\pi}{2}$	− 1
300°	$\frac{5\pi}{3}$	− .86
330°	$\frac{11\pi}{6}$	− .5
360°	2π	0

If $x =$	or	Then $y =$
0°	0	0
− 30°	$-\frac{\pi}{6}$	− .5
− 60°	$-\frac{\pi}{3}$	− .86
− 90°	$-\frac{\pi}{2}$	− 1
− 120°	$-\frac{2\pi}{3}$	− .86
− 150°	$-\frac{5\pi}{6}$	− .5
− 180°	$-\pi$	0
− 210°	$-\frac{7\pi}{6}$	.5
− 240°	$-\frac{4\pi}{3}$	.86
− 270°	$-\frac{3\pi}{2}$	1
− 300°	$-\frac{5\pi}{3}$	.86
− 330°	$-\frac{11\pi}{6}$	.5
− 360°	-2π	0

Plotting these points, we have the graph below:



Observations from the graph.

1. From the graph we can read off the variations of $\sin x$ as x changes. When x increases from $0°$ to $90°$, the value of $\sin x$ increases from ? to ? . When x increases from $90°$ to $270°$, the value of $\sin x$ decreases from ? to ? . What happens when x increases from $270°$ to $360°$? Trace the value of $\sin x$ as x decreases from $0°$ to $-360°$.

2. The graph also shows that the curve repeats itself after each interval or period of $360°$. Thus the sine is called a *periodic function* and has a *period* of $360°$ or 2π.

EXERCISES

1. What are the maximum and minimum values of $\sin x$?

2. Since the graph crosses the x-axis an infinite number of times, the equation $\sin x = 0$ has an infinite number of real roots, therefore $x =$? , ? , ? , ? , etc.

3. Discuss the variations of the function $\sin x$ as x increases from $0°$ to $360°$.

4. From the graph of $y = \sin x$, determine approximately the values of: (*a*) $\sin 10°$; (*b*) $\sin 20°$; (*c*) $\sin 80°$; (*d*) $\sin -40°$.

5. Determine approximately from the graph four angles whose sine is .4; whose sine is $-.6$.

6. Show by means of the graph that the sine of an angle is never greater than $+1$ and never less than -1.

7. Show from the graph that $\sin(180° - x) = \sin x$.

8. Show from the graph that $\sin(180° + x) = -\sin x$.

9. Draw the graph of and state the period for:

(*a*) $y = \sin 2x$. (*c*) $\sin^2 x$.

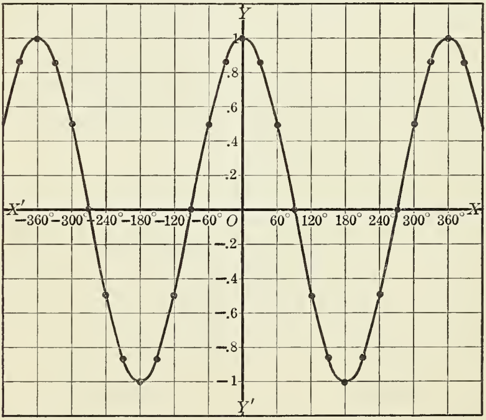
(*b*) $y = 2\sin x$. (*d*) $\sin \frac{x}{2}$.

The graph of $y = \cos x$. Setting up the tables as before, we have:

If $x =$	or $x =$	Then $y =$
0°	0	1
30°	$\frac{\pi}{6}$	.86
60°	$\frac{\pi}{3}$	.5
90°	$\frac{\pi}{2}$	0
120°	$\frac{2\pi}{3}$	$-.5$
150°	$\frac{5\pi}{6}$	$-.86$
180°	π	-1
210°	$\frac{7\pi}{6}$	$-.86$
240°	$\frac{4\pi}{3}$	$-.5$
270°	$\frac{3\pi}{2}$	0
300°	$\frac{5\pi}{3}$	.5
330°	$\frac{11\pi}{6}$	.86
360°	2π	1

If $x =$	or $x =$	Then $y =$
0°	0	1
$-30°$	$-\frac{\pi}{6}$	.86
$-60°$	$-\frac{\pi}{3}$	.5
$-90°$	$-\frac{\pi}{2}$	0
$-120°$	$-\frac{2\pi}{3}$	$-.5$
$-150°$	$-\frac{5\pi}{6}$	$-.86$
$-180°$	$-\pi$	-1
$-210°$	$-\frac{7\pi}{6}$	$-.86$
$-240°$	$-\frac{4\pi}{3}$	$-.5$
$-270°$	$-\frac{3\pi}{2}$	0
$-300°$	$-\frac{5\pi}{3}$	.5
$-330°$	$-\frac{11\pi}{6}$	.86
$-360°$	-2π	1

Plotting these points, we have the graph below:



Observations from the graph.

1. From the graph we can read off the variations of $\cos x$ as x changes. When x increases from $0°$ to $90°$, the value of $\cos x$ decreases from ? to ? .

2. The period of $\cos x$ is ? .

3. The cosine curve has the same shape as the sine curve and differs only in position.

EXERCISES

1. What are the maximum and minimum values of $\cos x$?

2. Discuss the variations of the function $\cos x$ as x increases from $0°$ to $360°$.

3. Plot the graphs of $\sin x$ and $\cos x$ on the same axes.

(*a*) Show from these graphs that $\cos x = \sin (90° - x)$.
(*b*) For what values of x will $\sin x = \cos x$?
(*c*) Find the approximate values of $\cos 10°$; $\cos 20°$; $\cos 80°$; $\cos (-40°)$.

4. Show from the graph that $\cos (180° + x) = -\cos x$.

5. Show from the graph that $\cos (180° - x) = -\cos x$.

6. Determine approximately from the graph:

(*a*) four angles whose cosine is .4.
(*b*) four angles whose cosine is $-.7$.

7. Show by means of the graph that the cosine of an angle is never greater than $+1$, and never less than -1.

8. Draw the graph of and state the period for:

(*a*) $y = \cos 2x$. (*b*) $2 \cos x$. (*c*) $\cos^2 x$. (*d*) $\cos \frac{x}{2}$.

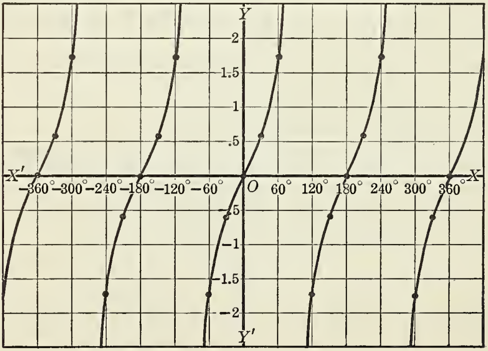
9. Draw the graph of $\sin x + \cos x$. Is this function periodic?

The graph of $y = \tan x$. Making the tables as explained previously, we have:

If $x =$	or $x =$	Then $y =$
0°	0	0
30°	$\frac{\pi}{6}$	.58
60°	$\frac{\pi}{3}$	1.73
90°	$\frac{\pi}{2}$	∞
120°	$\frac{2\pi}{3}$	-1.73
150°	$\frac{5\pi}{6}$	$-.58$
180°	π	0
210°	$\frac{7\pi}{6}$	.58
240°	$\frac{4\pi}{3}$	1.73
270°	$\frac{3\pi}{2}$	∞
300°	$\frac{5\pi}{3}$	-1.73
330°	$\frac{11\pi}{6}$	$-.58$
360°	2π	0

If $x =$	or $x =$	Then $y =$
0°	0	0
$-30°$	$-\frac{\pi}{6}$	$-.58$
$-60°$	$-\frac{\pi}{3}$	-1.73
$-90°$	$-\frac{\pi}{2}$	∞
$-120°$	$-\frac{2\pi}{3}$	1.73
$-150°$	$-\frac{5\pi}{6}$	.58
$-180°$	$-\pi$	0
$-210°$	$-\frac{7\pi}{6}$	$-.58$
$-240°$	$-\frac{4\pi}{3}$	-1.73
$-270°$	$-\frac{3\pi}{2}$	∞
$-300°$	$-\frac{5\pi}{3}$	1.73
$-330°$	$-\frac{11\pi}{6}$	.58
$-360°$	-2π	0

Plotting these points, we have the graph below:



Observe that the curve is discontinuous, the first "break" occurring when $x = 90°$ or $\frac{\pi}{2}$. Also the value of tan x changes sign, when it passes through the infinite values corresponding to $x = 90°$, and $x = 270°$, etc.

EXERCISES

1. The period of tan x is ? .

2. Discuss the variations of the function tan x as x increases from 0° to 360°.

3. Show from the graph that

(a) $\tan(180° - x) = -\tan x$. (b) $\tan(180° + x) = \tan x$.

4. Determine, approximately, from the graph on the opposite page, (a) four angles whose tangent is + 1; (b) four angles whose tangent is − 2.

5. Draw the graph of and state the period for:

(a) $y = \tan 2x$. (c) $\tan^2 x$.

(b) $y = 2\tan x$. (d) $\tan \frac{x}{2}$.

6. Does a tangent curve have maximum or minimum points?

7. Determine from the graph of $y = \tan x$ the approximate values of tan 10°; tan 20°; tan 80°; tan (− 40°).

The graphs of the reciprocal functions: csc x, sec x, cot x. Using the method of the previous paragraphs, the graphs of the cosecant, secant, and cotangent are readily obtained.

EXERCISES

1. Draw the graph of and state the period for:

(a) $y = \csc x$. (b) $y = \sec x$. (c) $y = \cot x$.

2. Show from the graph in 1 (c) that:

(a) $\cot(180° + x) = \cot x$.

(b) $\cot(180° - x) = -\cot x$.

3. Draw the graph of $y = \cot 2x$.

4. Draw the graph of $\cot x - \tan x$ from $x = 0$ to $x = \pi$.

5. Draw the graph of $y = \tan x + \sec x$.

6. Draw the graph of $y = \tan x + \cot x$.

7. Plot the graphs of $\tan x$ and $\cot x$ on the same axes and tell what the figure reveals about the relation between these functions.

A Further Study of Variation

Problem. The illumination from a source of light varies inversely as the square of the distance from the source. If my book is 8 feet from a lamp, at what distance from the lamp must I place it, in order that the light on it shall be doubled?

In order to solve this problem, it will first be necessary to learn the meaning and use of the new terms involved.

We have seen that the equation $xy = 12$ has for its graph an hyperbola, and that as x increases in value, the value of y decreases. This is precisely the " inverse " of the condition in the equation $y = 12\,x$, where as x increases, y increases.

When two variable quantities are so related that their product is constant, it is said that either one of the quantities *varies inversely* as the other or that either one is *inversely proportional* to the other.

Thus, in general, if $xy = K$, where K is a constant, then y varies inversely as x, or y is inversely proportional to x. Solving this equation for y, the relation $xy = K$ may be written in the form $y = \frac{K}{x}$.

Illustrative example. Express the following statement in equation form: The volume (V) of a gas varies inversely as the pressure (P).

Solution

$$V = \frac{K}{P} \text{ or } PV = K.$$

EXERCISES

Express each of the following statements in equation form:

1. The number of days (D) required to finish a job, other conditions being equal, varies inversely as the number of men (M) employed.

2. An electric current (I) varies inversely as the resistance (R).

3. The number of revolutions (R) of a wheel of a car varies inversely as the circumference (c) of the wheel.

4. The wave length (w) of a broadcasting station varies inversely as its number of kilocycles (k).

5. The tangent (tan) of an angle varies inversely as the cotangent (cot) of the same angle.

6. The cosine (cos) of an angle varies inversely as the secant (sec) of the same angle.

7. The cosecant (csc) of an angle varies inversely as the sine (sin) of the same angle.

8. The amount of heat (H) received from a heater varies inversely as the square of the distance (D) from it.

9. The intensity (T) of light on an object varies inversely as the square of the distance (D) from the source of light to the object.

10. The number (N) of vibrations made by a pendulum in one second varies inversely as the square root of its length.

Now to return to the problem at the beginning of this article as an:

Illustrative example. The illumination from a source of light varies inversely as the square of the distance from the source. If my book is 8 feet from a lamp, at what distance from the lamp must I place it, in order that the light on it shall be doubled?

Solution

Since the illumination (I) varies inversely as the square of the distance (D), then $I = \frac{K}{D^2}$. Now since the problem states that my book is 8 feet from the lamp, then $D = 8$.

Therefore $I = \frac{K}{8^2}$ or $I = \frac{K}{64}$, *i.e.*, $64\,I = K$.

Now we want to find D, in order that the illumination shall be doubled, that is, $2 \times I$ or $2\,I$.

Substituting $2\,I$ for I and $64\,I$ for K in the equation $I = \frac{K}{D^2}$, we have

$$2\,I = \frac{64\,I}{D^2}.$$

$$\therefore\ 2\,D^2I = 64\,I.$$

$$D^2 = 32.$$

$$D = \sqrt{32} = 4\sqrt{2}.$$

I should place the book about 5.6 feet from the lamp.

Illustrative example. If the volume of a gas varies inversely as the pressure, what is the volume under a pressure of 50 pounds when the gas occupies 36 cubic centimeters at a pressure of 15 pounds?

Solution

$$V = \frac{K}{P}.$$

Now since $V = 36$, when $P = 15$, substituting in the equation $V = \frac{K}{P}$, we have $36 = \frac{K}{15}$ or $K = 540$. Substituting the value, we have the formula $V = \frac{540}{P}$.

Then, when $P = 50$, $V = \frac{540}{50} = 10\frac{4}{5}$.

The volume of the gas under a pressure of 50 pounds is $10\frac{4}{5}$ cubic centimeters.

EXERCISES

1. The number of days required to complete a certain piece of work varies inversely as the number of men employed. If 10 men can complete the job in 45 days, how many men would be necessary to complete the same work in 15 days?

2. According to Boyle's Law the volume of a gas is inversely proportional to the pressure on it. A tank contains 4 cubic feet of air under a pressure of 60 pounds per square inch. How much space will be occupied by the air under an atmospheric pressure of 15 pounds per square inch?

3. The intensity of light on an object varies inversely as the square of the distance from the source of light to the object. An object is 15 feet from a lamp. At what distance from the lamp must the object be placed so that the intensity will be 9 times as great?

4. The intensity of light on an object varies inversely as the square of the distance from the source of light. If a book is 6 feet directly below a light, how will the illumination on the book change when the light is lowered 2 feet?

5. If x varies inversely as y, find x when:

(*a*) $y = 3$, if $x = 9$ when $y = 2$.
(*b*) $y = 15$, if $x = 3$ when $y = 10$.
(*c*) $y = -10$, if $x = 12$ when $y = 5$.

6. If y varies inversely as x, find y when:

(*a*) $x = 20$, if $y = 12$ when $x = 7$.
(*b*) $x = 8$, if $y = \frac{2}{3}$ when $x = 6$.
(*c*) $x = 5$, if $y = 20$ when $x = 4$.

Joint variation. If the formula for the area of a triangle $A = bh$ is considered carefully, it is observed that an increase in b will effect a corresponding increase in A, provided h remains constant. Similarly, an increase in h will cause an increase in A, provided b remains constant. For example, if b is doubled and h remains constant, then A will be doubled; if h is trebled and b remains constant, then A will be trebled. If, however, b is doubled and at the same time h is trebled, then A will be multiplied by 2×3 or 6. When a quantity varies as the product of several variables, it is said to *vary jointly* as the variables. Thus, in general, if x varies jointly as m, n, and p, then $x = Kmnp$, where K is a constant.

Illustrative examples.

Example 1. The volume of a circular cone varies jointly as its height and the square of the radius of its base. If a cone 30 inches high with a base whose radius is 4 inches has a volume of $160\,\pi$ cubic inches, what is the volume of a cone one-half as high with a radius twice as long?

Solution

$V = Khr^2$. Now $V = 160\,\pi$, when $h = 30$ and $r = 4$; substituting in the formula $V = Khr^2$, we have

$$160\,\pi = K\,30\,(4)^2.$$
$$160\,\pi = K\,480.$$
$$\frac{\pi}{3} = K.$$

Now if the cone is half as high and the radius is doubled, then $h = 15$ and $r = 8$. Substituting $\frac{\pi}{3}$ for K, 15 for h, and 8 for r in the formula $V = Khr^2$, we have

$$V = \frac{\pi}{3}(15)(8)^2 = 320\,\pi.$$

Combined variation. Frequently, however, the formula $A = bh$ may be written in the form $b = \frac{A}{h}\cdot$ An inspection of this equation shows that b varies directly as A (h remaining constant), and inversely as h (A remaining constant). This type of variation is frequently termed *combined* variation. In this case $b = \frac{KA}{h}\cdot$

Example 2. The time required for an automobile trip varies directly as the distance and inversely as the speed. If a trip of 250 miles takes 8 hours 20 minutes, how long will it take to travel 390 miles?

Solution

Since the time (t) varies directly as the distance (d) and inversely as the speed (r), then $t = \frac{Kd}{r}$.

Now $d = 250$ when t is $8\frac{1}{3}$.

Substituting in the equation $t = \frac{Kd}{r}$,

$$8\tfrac{1}{3} = \frac{K\,250}{r}$$

$$K = \frac{8\frac{1}{3}\,r}{250} = \frac{r}{30}.$$

But we wish to know t when d is 390.

Substituting $\frac{r}{30}$ for K, and 390 for d in the formula $t = \frac{Kd}{r}$,

$$t = \frac{\frac{r}{30} \times 390}{r} = 13 \text{ hours.}$$

EXERCISES

1. The volume of a circular cylinder varies as the height and as the square of the radius of the base. What must be the height of a cylinder with a 4-inch radius, if it is to contain 8 times as much as another cylinder 5 inches high and having a radius of 3 inches?

2. The area of the curved surface of a right circular cylinder varies jointly as the radius and the altitude. If a right circular cylinder having a radius 4 and altitude 10, has an area $80\,\pi$, what is the area of the curved surface of a right circular cylinder whose radius is 5 and height 8?

3. If x varies jointly as y and z, and $x = 36$ when $y = 9$ and $z = 1$, find x when $y = 3$ and $z = 3$.

4. The time required to repair a sewer varies directly as the length and inversely as the number of men employed to do the job. If it takes 10 men 3 days to repair a sewer 50 feet long, how long will it take 9 men to repair 150 feet of the same type of sewer?

5. If y varies directly as x, and inversely as $\sqrt{a}$, and $y = 6$ when $x = 6$ and $a = 9$, find the value of y when $x = 4$ and $a = 25$.

Getting to the Roots

Early writers on mathematics had considerable difficulty in understanding the solution of the quadratic equation, due to the fact that negative or irrational numbers were not understood. However, these equations were solved by the Egyptians arithmetically and later by Euclid geometrically. Diophantus (about 250 A.D.) solved quadratic equations by a method very similar to that of completing the square. However, he avoided equations with negative or irrational roots, even going so far as to reject the whole problem as impossible, when arriving at such results. He also failed to observe that a quadratic equation has two roots. To Sridhara (about 1020 A.D.) is attributed the "Hindu Method" of completing the square by multiplying both sides of the equation by 4 times the coefficient of x^2, adding the square of the original coefficient of x to both sides, and extracting the square root.

CUMULATIVE REVIEW

Chapters VI, VII, and VIII

1. Which of these statements are true? Which are false?

(*a*) By the law of the lever objects of unequal weights can be balanced.

(*b*) If the area of a square is $4s^2$, the perimeter of the square is $4s$.

(*c*) If a point is 10 units distant from the origin, then the sum of the squares of the coördinates of the point is 100.

(*d*) There is no quadrant in which the tangent and cotangent of an angle have opposite signs.

(*e*) If $\tan x = -1$ and x is in Q III, $x = -225°$.

(*f*) The graph of every linear equation is a straight line.

(*g*) The x- and y-intercepts of the graph of $3x = y + 3$ are equal.

(*h*) The relation $A = Kr^2$ signifies that A varies directly as the square of r.

(*i*) The graph of $A = \frac{22}{7} r^2$ is a parabola.

(*j*) The graph of a parabola is symmetrical with respect to the two axes.

2. Complete each of the following statements:

(*a*) The formula which expresses the relation between x and y as suggested in the table:

x	1	2	3	7	10
y	-6	-5	-4	0	3

is ? .

(*b*) If the radius of a circle is doubled, the area of the circle is multiplied by ? .

(*c*) As angle x changes from 180° to 90°, $\tan x$ changes from ? to ? .

(*d*) If x is an acute angle, the function that can increase from $\frac{1}{3}$ to 2 as x increases is ? .

(*e*) $\tan 120° = -\cot$? .

(*f*) The value of the tangent of an angle depends upon the size of the angle. Therefore the ? is a function of the ? .

(*g*) If the graph of an equation is to pass through the origin, the constant term must be ? .

(*h*) The graph of $x^2 + 4y^2 = 9$ is an ? .

(*i*) In order that $x^2 - 5x$ may be the first two terms of a perfect square trinomial add ? .

(*j*) The discriminant of the equation $x^2 = 25$ is ? .

3. A crowbar 5 feet long is used to lift a weight of 325 pounds. If the fulcrum is placed 8 inches from the weight, calculate in pounds the effort needed at the other end of the crowbar to raise the weight.

4. (*a*) A boy opens a school bank account by depositing \$5. If each week thereafter he deposits 25¢, write a formula for the amount of money (A) he will have in the bank at the end of w weeks.

(*b*) Using this formula, calculate the amount of his savings at the end of a year.

5. Trace the changes, both in sign and in numerical value, in the sine of an angle θ as θ changes continuously from 90° to 360°.

6. Construct the line functions of a negative angle in Q IV. Label each line by the function it represents and indicate its sign.

7. Solve for the positive values of x less than 360° and check: $2 \sin^2 x + 7 \cos x = 2$.

8. Write the equation of a line whose slope is $-\frac{1}{3}$ and whose y-intercept is 5.

9. The distance (in miles) an object is visible at sea varies as the square root of the height of the object (in feet) above sea level. If from a lighthouse 36 feet high ships can be seen at a maximum distance of 7.32 miles, what is the greatest visible distance from the top of a lighthouse 49 feet high?

10. (*a*) Form a table of values for $y = x^2 - 3x$ by assigning to x all integral values from -1 to 4 inclusive.

(*b*) Using this table, draw the graph of $y = x^2 - 3x$.

(*c*) From the graph, estimate, correct to the nearest tenth, the roots of $x^2 - 3x = 2$.

11. Construct the graph of $xy = 10$.

12. Construct the graph of $y = \frac{1}{2} \sin x$ and state its period.

CHAPTER IX. SYSTEMS OF EQUATIONS

What science can there be more noble, more excellent, more useful for men, more admirably right and demonstrative, than this of mathematics?

— Benjamin Franklin.

Systems of Linear Equations

As you know, it is advisable to use two unknowns instead of one in solving certain problems. We shall now recall some of the work you have had, before we attempt to solve such problems.

Problem. The Economy Grocery Store sells 6 lb. of sugar with 1 lb. of tea for 75 cents, or 1 lb. of sugar with 2 lb. of tea for 95 cents. At these rates, what are their prices for tea and sugar?

If we let x = the cost in cents of 1 lb. of sugar,
and y = the cost in cents of 1 lb. of tea,

we may write two linear equations that meet the conditions stated in the problem:

$$6x + y = 75.$$
$$x + 2y = 95.$$

This pair of linear equations may be solved by:

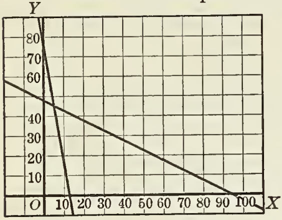
(*a*) the graphic method.
(*b*) the addition and subtraction method.
(*c*) the substitution method.

To appreciate the ease with which these equations were formed because two unknowns were used, let us form a single equation using only one unknown. If x = the cost in cents of 1 lb. of tea, then $\frac{75 - x}{6}$ = the cost in cents of 1 lb. of sugar. Why?

Then $\frac{75 - x}{6} + 2x = 95.$ Why?

The graphic method of solving a pair of linear equations. The graphs of $6x + y = 75$ and $x + 2y = 95$ are shown in the diagram below.

The following questions will help you to recall what you have learned about pairs of linear equations.



1. How many pairs of values of x and y satisfy the equation $6x + y = 75$?

2. How many pairs of values of x and y satisfy the equation $x + 2y = 95$?

3. How many pairs of values of x and y satisfy, *i.e.*, are common to, both equations? How is this fact shown on the graph?

4. What is the value of x and the value of y which satisfy both equations?

5. What fact in geometry tells you that there cannot be more than one point in common, *i.e.*, one set of answers?

When a system of equations has a common solution, the equations are known as *simultaneous equations*.

EXERCISES

Solve graphically:

1. $x + y = 5$
$x - y = 3.$

2. $x - 2y = 1$
$x + y = 7.$

3. $2x + y = 5$
$3x - 2y = 18.$

4. $4x + 5y = 24.$
$3x - 2y = -5.$

5. $2x + y = 9$
$x = 4y.$

6. $3x - 5y = 5$
$2y = x - 1.$

Something to think about.

1. (*a*) Graph the equations $x + 2y = 4$ and $x + 2y = 6$. What is the relation between the two graphs?

(*b*) Have the equations a common solution?

(*c*) Find the slope of the graph of each equation and tell what relation exists between the slopes.

(*d*) From a study of the two equations it is seen that if one of them is true, the other is not. Why?

When a system of equations does not have a common solution, the equations are said to be *inconsistent* equations.

2. (*a*) Graph the equations $x + 2y = 4$ and $2x + 4y = 8$.

(*b*) What is the relation between the two graphs?

(*c*) Have they a common solution?

(*d*) The second equation may be derived from the first by multiplying it by 2, consequently any pair of numbers that satisfies the first equation will satisfy the second. Why?

In such a system, the equations are called *dependent*. All other sets are called *independent*.

EXERCISES

Are the following equations inconsistent, dependent or independent?

1. $2x + 3y = 7$
$4x + 6y = 14.$

2. $3x + 2y = 8$
$3x + 2y = 10.$

3. $x + 2y = 11$
$2x + y = 11.$

4. $2x + 5y = 31$
$5y + 2x = 17.$

5. $2x - 3y = 12$
$x - 1.5y = 6.$

6. $3x + 8y = 14$
$3x + 8y = 4.$

The algebraic method of solving a pair of linear equations. The graphic method of solving simultaneous equations is not always convenient and is not accurate if the roots are fractions or large numbers. Therefore, we resort to algebraic methods.

To solve two equations in two unknowns we must first obtain one equation containing only one unknown. This process is called *elimination*. Two methods of elimination will be considered and reviewed.

Recall fact 46. *To eliminate one unknown from a pair of linear equations by addition and subtraction, multiply each equation by such a number that the coefficients (without regard to sign) of one unknown become the same in both equations, and then add or subtract the equations.*

Illustrative example. Solve $3x + 4y = 17$ (1)

$$5x + 6y = 27. \qquad (2)$$

Analysis

To eliminate y, multiply (1) by 3 and (2) by 2, and then subtract the resulting equations.

Solution

$$\begin{array}{rl} 9x + 12y = & 51 \\ \underline{10x + 12y =} & \underline{54} \\ -x \qquad\quad = & -3 \\ x = & 3. \end{array}$$

Substituting 3 for x in (1),

$$9 + 4y = 17.$$
$$y = 2.$$

Answer: $x = 3$ and $y = 2$.

Check

Substituting in (1),

$$3x + 4y = 17.$$
$$9 + 8 = 17.$$
$$17 = 17.$$

Substituting in (2),

$$5x + 6y = 27.$$
$$15 + 12 = 27.$$
$$27 = 27.$$

EXERCISES

Solve by the addition and subtraction method and check :

1. $x + 3y = 9$
   $2x - y = 4.$
2. $2x - y = 3$
   $3x - 4y = -8.$
3. $2x + 4y = 17$
   $x - y = 1.$
4. $2x - 3y + 5 = 0$
   $x + 2y = 2.$
5. $\frac{x}{3} + \frac{y}{4} = 6$
   $\frac{x}{4} + \frac{y}{2} = 7.$

6. $\frac{x}{6} - \frac{y}{3} = 1$
$3x + 5y = 9.$

7. $\frac{4y+1}{2} - \frac{3x+1}{3} = \frac{1}{2}$
$\frac{2x-1}{2} - \frac{4y-1}{3} = \frac{1}{2}.$

8. $\frac{3}{x-1} + \frac{1}{y-4} = 0$
$\frac{5}{x+1} - \frac{4}{y+1} = 0.$

9. $\frac{x+4}{y+1} = 2$
$\frac{x+2}{y-1} = 3.$

10. $\frac{4}{x} + \frac{3}{y} = 2$
$\frac{8}{x} - \frac{3}{y} = 1.$

11. $\frac{3}{x} - \frac{2}{y} = 0$
$\frac{6}{x} + \frac{4}{y} = 4.$

Recall fact 47. *To eliminate one unknown from a pair of linear equations by substitution, find the value of either unknown in terms of the other in one of the equations and substitute this value in the other equation.*

Illustrative example. Solve the equations

$$3x + 4y = 17 \quad (1)$$

$$5x + 6y = 27. \quad (2)$$

Solution

Solving (1) for x in terms of y, we obtain

$$x = \frac{17 - 4y}{3}. \quad (3)$$

Substituting this value of x in (2),

$$5\left(\frac{17-4y}{3}\right) + 6y = 27.$$

$$-2y = -4.$$

$$y = 2.$$

Substituting 2 for y in equation (3),

$$x = \frac{17-8}{3} = \frac{9}{3} = 3.$$

Answer: $x = 3$ and $y = 2$.

Check by substituting $x = 3$ and $y = 2$ in both equations.

EXERCISES

Solve and check each of the following sets of equations by the substitution method.

1. $x + y = 11$
   $x - y = -3.$
2. $x + y = 6$
   $3x + 2y = 13.$
3. $y = 2x + 1$
   $y = 3x - 5.$
4. $x = 4y$
   $x + 4y = 80.$
5. $3a - 2b = 5$
   $a + b = 27.$
6. $2x - 3y + 5 = 0$
   $x + 2y = 2.$
7. $5x + 3y = 1$
   $2x + y = -4.$
8. $k + 3l = -6$
   $2k - 4l = -12.$
9. $4m + 3n = 11$
   $3m - 4n = 2.$
10. $4x = 3y$
    $2x - 5y = -7.$
11. $\dfrac{2y - 6}{5} = x$
    $y - x = 9.$
12. $\dfrac{7x - 15}{3} = y$
    $2x - y = 3.$
13. $x = 8y + 1$
    $y = \frac{1}{4}x - 2.$

Simultaneous literal equations. Systems of linear equations of this kind may be solved conveniently by either of the algebraic methods.

Illustrative example. Solve

$$mx - ny = m^2 + n^2 \qquad (1)$$
$$x - y = 2n. \qquad (2)$$

Solution

To eliminate x, multiply (2) by m.

$$\begin{array}{r|l} 1 & mx - ny = m^2 + n^2 \\ m & \quad x - y = 2n \end{array}$$

$$\begin{array}{lrcl} & mx - ny &=& m^2 + n^2 \\ & mx - my &=& 2mn \\ \hline \text{Subtracting} & my - ny &=& m^2 - 2mn + n^2 \\ \text{Factoring} & y(m - n) &=& (m - n)(m - n). \\ & y &=& m - n. \end{array}$$

Substituting $(m - n)$ for y in (2),

$$x - (m - n) = 2n.$$
$$x - m + n = 2n.$$
$$x = m + n.$$

Answer: $x = m + n$ and $y = m - n$.

Check

Substituting in (1)

$$mx - ny = m^2 + n^2.$$
$$m(m + n) - n(m - n) = m^2 + n^2.$$
$$m^2 + mn - mn + n^2 = m^2 + n^2.$$
$$m^2 + n^2 = m^2 + n^2.$$

Substituting in (2)

$$(m + n) - (m - n) = 2n.$$
$$m + n - m + n = 2n.$$
$$2n = 2n.$$

EXERCISES

Solve for x and y:

1. $bx + ay = 2ab$
   $ax + by = a^2 + b^2.$
2. $2x + y = a + 1$
   $x - 2y = a - 1.$
3. $mx - ny = m^2 - n^2$
   $nx - my = 0.$
4. $Lx + y = -D$
   $x + Ly = D.$
5. $ax + by = c$
   $bx - ay = d.$
6. $y = x \tan 30°$
   $y = (x + 200) \tan 60°.$
7. $\dfrac{50 + y}{x} = \tan 45°$
   $\dfrac{50}{x} = \tan 30°.$
8. $\dfrac{x}{y} = \tan 45°$
   $\dfrac{x}{y - 200} = \tan 60°.$
9. $\dfrac{25}{x} = \tan 30°$
   $\dfrac{25 + y}{x} = \tan 45°.$

10. Eliminate t from the formulas $v = at$ and $s = \frac{1}{2} at^2$ and solve for v.

11. Using the formulas $l = a + (n - 1)d$ and $s = \dfrac{n}{2}(a + l)$, write a formula for s which does not contain a.

12. Using the formulas $C = 2\pi r$ and $K = \pi r^2$, write a formula for K in terms of C.

13. Eliminate E from the formulas $C = \frac{E}{R}$ and $H = \frac{E^2t}{R}$ and solve for H.

14. Using the formulas $P = A - i$ and $i = Prt$, write a formula for P in terms of A, r, and t.

15. Eliminate $\cos A$ from the formulas $a^2 = b^2 + c^2 - 2bc \cos A$ and $\cos A = \frac{p}{c}$, and then solve the resulting formula for p.

Using simultaneous linear equations to find the equation of a line. Very frequently it is necessary for the scientist to find the equation of a line from the results of experiments. We shall now study the method he employs.

Problem. Find the equation of a line passing through the points (2, 4) and (3, 5).

The general equation of a straight line is $y = mx + b$. Now since the first point (2, 4) lies on this line, $x = 2$ and $y = 4$ must satisfy this equation, *i.e.*, $4 = 2m + b$. (1)

Again, since the second point (3, 5) lies on this line, $x = 3$ and $y = 5$ must satisfy this equation, *i.e.*, $5 = 3m + b$. (2)

We thus have two equations, (1) and (2), involving two unknowns, m and b. Let us solve these two equations:

$$4 = 2m + b \qquad (1)$$
$$5 = 3m + b \qquad (2)$$

Subtracting
$$-1 = -m.$$
$$\therefore 1 = m.$$

Substituting 1 for m in equation (1),

$$4 = 2 + b.$$
$$\therefore 2 = b.$$

Substituting the values for m and b in the general equation $y = mx + b$, we have
$$y = x + 2.$$

This is the equation of the line which passes through the given points.

Something to think about.

1. Does the point (5, 7) lie on the line $y = x + 2$? Why?
2. Does the point (3, 6) lie on the line $y = x + 2$? Why?

EXERCISES

Group A

Find the equation of the line passing through:

1. (1, 2) and (4, 5).

2. (−2, 1) and (3, − 4).

3. (− 2, − 1) and (5, 3).

4. (−4, −1) and (−6, 2).

5. (−3, −2) and (−5, −7).

6. (0, 5) and (3, 0).

7. (0, 0) and (− 2, 5).

8. $(7, 3\frac{1}{2})$ and $(3, -2\frac{1}{4})$.

Group B

9. The coördinates of the vertices of a triangle are (3, 5), (1, 3), and (5, 2). Find the equation for each side of the triangle.

The length of a line joining two points is given by the formula:

$$L = \sqrt{(\text{Difference between abscissas})^2 + (\text{Difference between ordinates})^2}.$$

Use this formula to:

10. Find the length of the line joining the points:
(1, 3) and (5, 6); (2, 4) and (3, 5); (− 1, 2) and (4, − 5).

Group C

11. Find the lengths of the sides of a triangle whose vertices are (4, 3), (10, 8), and (2, 5).

12. Show that the points (3, 3), (9, 3), and (6, 8) are the vertices of an isosceles triangle.

13. Prove that the points (3, 3), (3, 0), and (8, 3) are the vertices of a right triangle.

14. Prove that the lines joining the points (2, 3), (11, 3), (15, 9), and (6, 9), in succession form a parallelogram.

Sets of linear equations involving three unknowns. The graphic method used to solve two linear equations in two unknowns is not possible with equations in three unknowns as the values of these three unknowns cannot be plotted in a plane. However, algebraic methods may be used.

Illustrative example.

Solve:

$$
\begin{aligned}
x - 2y + 3z &= 6 &\quad (1)\\
2x + 3y - 4z &= 20 &\quad (2)\\
3x - 2y + 5z &= 26. &\quad (3)
\end{aligned}
$$

Analysis

Eliminate one unknown from any pair of equations and the same unknown from another pair. In this example eliminate x from (1) and (2) and also from (1) and (3).

Solution

$$
\begin{array}{r|rl}
2 & x - 2y + 3z = 6 & (1)\\
1 & 2x + 3y - 4z = 20 & (2)\\
\hline
 & 2x - 4y + 6z = 12 & \\
 & 2x + 3y - 4z = 20 & \\
\hline
 & -7y + 10z = -8 & (4)
\end{array}
$$

$$
\begin{array}{r|rl}
3 & x - 2y + 3z = 6 & (1)\\
1 & 3x - 2y + 5z = 26 & (3)\\
\hline
 & 3x - 6y + 9z = 18 & \\
 & 3x - 2y + 5z = 26 & \\
\hline
 & -4y + 4z = -8 & \\
 & -y + z = -2 & (5)
\end{array}
$$

The problem has now been reduced to that of solving two equations in two unknowns. We can solve these two equations (4) and (5) as follows.

$$
\begin{array}{r|rl}
1 & -7y + 10z = -8 & (4)\\
7 & -y + z = -2 & (5)\\
\hline
 & -7y + 10z = -8 & \\
 & -7y + 7z = -14 & \\
\hline
 & 3z = 6 & \\
 & z = 2. &
\end{array}
$$

Substituting 2 for z in (4)

$$
\begin{aligned}
-7y + 20 &= -8.\\
-7y &= -28.\\
y &= 4.
\end{aligned}
$$

Substituting 2 for z and 4 for y in equation (1), we have

$$
\begin{aligned}
x - 8 + 6 &= 6.\\
x &= 8.
\end{aligned}
$$

Answer: $x = 8$, $y = 4$, and $z = 2$.

Check by substituting in each of the three given equations.

EXERCISES

Solve and check:

1. $2x - 2y + 5z = 14$
   $3x + y + 3z = 8$
   $5x - 5y + 2z = 14.$
2. $3x + 2y - z = 8$
   $2x + 3y + 6z = 2$
   $x - y - 5z = 4.$
3. $2x - 3y - z = 6$
   $3x + 5y + 4z = 5$
   $x + 2y + z = 1.$
4. $x + 2y + z = 12$
   $2x - y + z = 5$
   $3x + y - 2z = 1.$
5. $2x + y - 3z = -5$
   $x + 3y = 7$
   $2z - 5y = -4.$
6. $2x + 3y = 3z$
   $z - 2y = 2$
   $x + z = 4.$
7. $x + y + z = 8$
   $x - 3z = 25$
   $2y + 5z + 41 = 0.$
8. $3x - y = 7$
   $3y - z = 5$
   $3z - x = 0.$

Using simultaneous linear equations to find the equation of a curve.

Illustrative example. Find the equation of a parabola passing through the points (0, 2), (1, − 1), and (− 3, 35).

Analysis

The equation of the parabola is $y = ax^2 + bx + c$. (1)
Since the points given lie on the parabola, their coördinates must satisfy equation (1).

Solution

∴ Substituting $x = 0, y = 2$	$2 = c.$
Substituting $x = 1, y = -1$	$-1 = a + b + c.$
Substituting $x = -3, y = 35$	$35 = 9a - 3b + c.$

We have now three equations with three unknowns. Solving these equations, we find that $a = 2$, $b = -5$, $c = 2$.

Substituting these values in equation (1),

$$y = 2x^2 - 5x + 2.$$

EXERCISES

Find the equation of the parabola passing through the points:

1. $(1, -4), (5, 12), (-2, 5)$.
2. $(2, 3), (1, -2), (-2, -5)$.
3. $(0, 0), (2, 8), (-3, 18)$.
4. $(0, -7), (3, 20), (-2, 5)$.

DIGIT PROBLEMS

Many interesting problems concerning the digits of a number can readily be solved by using the method of simultaneous equations.

Recall fact 48. *The number* 24 *is a short way of writing* 2 *tens plus* 4 *units or* $20 + 4$. *In general, if* t *represents the tens' digit and* u *the units' digit of a number, the number is represented by* $10t + u$.

Group A

If t and u represent the tens' digit and the units' digit respectively of a two-digit number, represent the following statements algebraically:

1. The original number.
2. The sum of the digits.
3. The number when the digits are interchanged, *i.e.*, reversed.
4. The tens' digit exceeds the units' digit by 6.
5. The number resulting when the digits are interchanged is 26.
6. One third of the number equals the sum of the digits.

If h, t, and u are the hundreds' digit, the tens' digit, and the units' digit respectively of a three-digit number, express the following statements algebraically:

7. The sum of the digits.
8. The number.

9. The number with the digits reversed.

10. The ratio of the number to the sum of its digits is 52 : 1.

Illustrative example. The sum of the digits of a two-digit number is 7. If the digits are interchanged, the resulting number is 27 less than the original number. Find the number.

Let t = the tens' digit, and u = the units' digit.

$$\begin{aligned} \text{Then } 10t + u &= \text{the number} \\ \text{and } 10u + t &= \text{the number with digits reversed} \\ \text{and } t + u &= \text{the sum of the digits.} \\ \text{Then } t + u &= 7 \qquad (1) \\ \text{and } (10u + t) &= (10t + u) - 27. \qquad (2) \end{aligned}$$

Simplifying (2) $\quad u - t = -3.$

	$t + u = 7$	Substituting 2 for u in (1)
	$-t + u = -3$	$t + 2 = 7.$
Adding	$2u = 4$	$t = 5.$
	$u = 2.$	The number is 52.

Check in the original problem.

11. The sum of the digits of a two-digit number is 8. If the digits are interchanged, the resulting number is 18 less than the original number. Find the number.

12. The sum of the digits of a certain number of two digits is 12, and the number is 2 less than 11 times the tens' digit. Find the number.

13. The sum of the digits of a two-digit number is 11. If the digits are interchanged, the new number will be 20 less than twice the original number. Find the number.

14. A number consisting of two digits is 3 times as large as the sum of the digits. If the digits be reversed and the resulting number added to the original number, the sum is 99. Find the number.

15. The units' digit of a two-digit number is twice the tens' digit. If the digits are interchanged, the number is increased by 9. Find the number.

16. The sum of the digits of a two-digit number is 11. If 9 be subtracted from the number, the resulting number has the same digits in reverse order. Find the number.

17. The units' digit of a certain number of two digits exceeds the tens' digit by 5. The sum of the digits is $\frac{1}{3}$ of the number. Find the number.

Group B

18. The units' digit of a two-digit number exceeds the tens' digit by 1. If the number is divided by the sum of the digits, the quotient is 5. Find the number.

19. The sum of the digits of a two-digit number is 8. If 10 is added to the number and the result divided by the sum of the digits, the quotient is 5 and the remainder 5. Find the number.

20. The sum of the three digits of a number is 18. The tens' digit is equal to the sum of the other two. If the digits are reversed in order, the resulting number is 99 less than the original number. Find the number.

21. The sum of the three digits of a number is 16 and the sum of the hundreds' digit and units' digit is equal to the tens' digit. If the units' digit and tens' digit change places, the resulting number will be 27 less than the original number. Find the original number.

Group C

22. A three-digit number can be written $100\,h + 10\,t + u$, and this can be rewritten $9(11\,h + t) + (h + t + u)$. From this fact prove that if the sum of the digits of a three digit number is divisible by 9, then the number itself is divisible by 9. Can this rule be extended beyond numbers of three digits?

23. Show that if the tens' digit of any two-digit number is twice the units' digit, then the number is always 7 times the sum of the digits.

24. Prove that the difference between any three-digit number and the sum of its digits is always divisible by 9.

FRACTION AND RATIO PROBLEMS

Group A

If x is the numerator and y the denominator of a fraction, write:

1. The fraction. **2.** The reciprocal of the fraction.

Write the fraction when:

3. 3 is subtracted from the numerator and 5 is added to the denominator.

4. The numerator is twice the denominator.

5. The numerator and denominator are in the ratio of 2 : 3.

6. Write the equation when the fraction plus its reciprocal is $2\frac{1}{2}$.

Illustrative example. If 4 is added to both the numerator and the denominator of a certain fraction, the resulting fraction equals $\frac{3}{4}$; if 2 is subtracted from both numerator and denominator, the fraction equals $\frac{1}{2}$. Find the original fraction.

Solution

Let x = the numerator and y = the denominator.

$$\text{Then } \frac{x}{y} = \text{the fraction.}$$

$$\text{Then } \frac{x+4}{y+4} = \frac{3}{4}$$

$$\text{and } \frac{x-2}{y-2} = \frac{1}{2}.$$

Solving $x = 5, y = 8$.

The fraction is $\frac{5}{8}$.

Check in the original problem.

7. If 3 is added to the numerator and 1 to the denominator of a certain fraction, the value becomes $\frac{1}{2}$. If 3 is subtracted from the numerator and denominator of the same fraction, its value becomes $\frac{1}{3}$. Find the fraction.

8. If 1 is added to the numerator of a fraction, the value of the fraction becomes $\frac{1}{2}$. If 1 is added to the denominator of the same fraction, the value becomes $\frac{1}{5}$. Find the fraction.

9. The numerator of a fraction is 7 less than the denominator; if 4 is subtracted from the numerator and 1 added to the denominator, the resulting fraction equals $\frac{1}{3}$. Find the fraction.

10. What number must be added to the numerator and denominator of the fraction $\frac{23}{35}$ so that the resulting fraction shall equal $\frac{3}{4}$?

11. The denominator of a certain fraction exceeds the numerator by 5, and $\frac{1}{2}$ the numerator plus $\frac{1}{3}$ the denominator is equal to 15. Find the fraction.

Group B

12. The numerator of a certain fraction is to its denominator as 2 : 3. Let $2x =$ the numerator and $3x =$ the denominator. Why? If 5 be added to the numerator, the ratio will be as 3 : 2. Find the fraction.

13. The numerator and denominator of a certain fraction are in the ratio of 2 : 3; if 3 is subtracted from the numerator and 6 from the denominator, the value of the resulting fraction is $\frac{3}{4}$. Find the original fraction.

14. The numerator and denominator of a certain fraction are in the ratio of 3 : 5. If 5 is subtracted from both the numerator and denominator, the ratio will be 1 : 2. Find the fraction.

15. Two numbers are in the ratio of 7 : 9. If 14 is added to each number, they will be in the ratio of 4 : 5. Find the numbers.

Group C

16. The sum of two fractions whose numerators are 3 is three times the smaller fraction; three times the smaller subtracted from twice the larger is $\frac{3}{8}$. Find the fractions.

17. What number must be subtracted from the numerator and the denominator of the fraction $\frac{c}{d}$ in order that the resulting fraction may have the value $\frac{a}{b}$?

MOTION PROBLEMS

You will remember from your work in elementary algebra that:

Rate downstream = rate in still water + rate of stream.
Rate upstream = rate in still water − rate of stream.

Illustrative example. A steamer goes downstream 48 miles in 3 hours. The return trip takes 1 hour longer. What is the rate of the stream and the rate of the steamer in still water?

Solution

$$\begin{aligned}
\text{Let } x &= \text{rate of the steamer in still water}\\
\text{and } y &= \text{rate of the stream.}\\
\text{Then } (x+y) &= \text{rate downstream}\\
\text{and } (x-y) &= \text{rate upstream.}\\
\text{Then } \frac{48}{x+y} &= 3\\
\text{and } \frac{48}{x-y} &= 4.\\
\text{Then } 48 &= 3x + 3y\\
\text{and } 48 &= 4x - 4y.\\
\therefore x &= 14 \text{ and } y = 2.
\end{aligned}$$

The rate of the steamer in still water is 14 miles per hour; the rate of the stream is 2 miles per hour.

Check by substituting in the statements of the original problem.

1. A man rowed 8 miles up a river in 4 hours and returned in 2 hours. Find the rate of the current and the man's rate of rowing in still water.

2. A boy rowed 10 miles down a river in $1\frac{2}{3}$ hours, and returned in 5 hours. Find the rate of the current and the rate of rowing in still water.

3. A boy can row 3 miles an hour in still water. He can row 9 miles upstream and back in 8 hours. What is the rate of the current?

4. The rate of the current of a river is $1\frac{1}{2}$ miles an hour. If a boat can go a certain distance downstream in 2 hours and return in $2\frac{1}{2}$ hours, what is the distance the boat went and the rate of the boat in still water?

5. An airplane flew a distance of 320 miles in 2 hours when going with the wind. Returning against the wind it was able to travel the same distance in 4 hours. Find the speed of the plane and the velocity of the wind.

6. A boat's crew can row downstream a distance of 10 miles in 50 minutes, and upstream a distance of 12 miles in 90 minutes. Find the rate of the current and the rate of the crew in still water.

7. A crew takes h hours to row a certain distance upstream. It takes them m hours to row back the same distance. If the rate of the crew in still water is r miles an hour, what is the rate of the stream?

8. An airmail pilot can fly 200 miles an hour in still air. On a stormy day he required $2\frac{2}{3}$ hours to fly a distance of 250 miles against the wind and back again. What was the velocity of the wind?

9. In Example 8, if the pilot's rate in still air is m miles per hour, t his time going and returning, and d the distance flown one way, write a formula for the velocity (v) of the wind in terms of m, t, and d.

BLENDING PROBLEMS

Illustrative example. Mr. Williams has coffee worth 52 cents a pound, but to undersell a rival he wishes to mix it with 36-cent coffee and sell the mixture at 44 cents a pound. To make a mixture of 60 pounds, how many pounds of each must he use?

Solution

Let x = number of pounds at 52¢ and y = number of pounds at 36¢.

Kind	Weight in Pounds	Cost	Value in Cents
First	x	52	$52x$
Second	y	36	$36y$
Mixture	60	44	2640 total

$$\begin{aligned} x + y &= 60 \\ 52x + 36y &= 2640 \\ \hline x &= 30 \text{ lb. at } 52¢ \\ y &= 30 \text{ lb. at } 36¢. \end{aligned}$$

Check in the original problem.

1. A grocer wishes to mix sugar at 8 cents a pound with sugar at 5 cents a pound. How many pounds of each must he use to make a mixture of 60 pounds worth 6 cents a pound?

2. A tea merchant has tea worth 65 cents a pound and also tea worth 50 cents a pound. He wishes to blend these teas so as to form a mixture of 12 pounds worth 60 cents a pound. How many pounds of each grade should he take?

3. A dealer has two kinds of coffee, worth 30 and 40 cents per pound. How many pounds of each must be taken to make a mixture of 70 pounds, worth 36 cents per pound?

4. How many pounds of a 60-cent grade tea and how many pounds of a 90-cent grade tea must be taken to make a mixture of 120 pounds worth 80 cents a pound?

5. Milk testing 3% butter fat is to be mixed with cream testing 17% butter fat. How many gallons of each must be used to make a mixture of 10 gallons which shall be 10% butter fat?

6. A dairyman wishes to combine milk and cream to make 40 gallons of a mixture which shall contain 25% butter fat. If the milk contains 5% butter fat and the cream 30% butter fat, how many gallons of each must he use?

7. A goldsmith has two alloys of gold, the first being 75% pure gold and the second $\frac{5}{12}$ pure gold. How much of each must be taken to make an alloy of 100 ounces which shall be $\frac{2}{3}$ pure gold?

8. Two casks contain each a mixture of wine and water. In the first the ratio of wine to water is 1 : 3, and in the second 3 : 5. How many gallons must be taken from each so as to form a mixture which shall contain 5 gallons of wine and 9 gallons of water?

9. A photographer has two bottles of developer. In one bottle 10% of the contents is developer and the rest water; in the other the mixture is half and half. How much must be drawn from each to make 8 ounces of a mixture in which 25% is developer?

Systems of Quadratic Equations

One quadratic and one linear equation. Certain problems lead to a system of equations such that one equation is quadratic and the other linear.

Problem. The sum of the areas of two square pieces of tin is 25 square feet and a side of the larger square is 1 foot longer than a side of the smaller. Find the length of a side of each square.

If we let x = number of feet in a side of the larger square

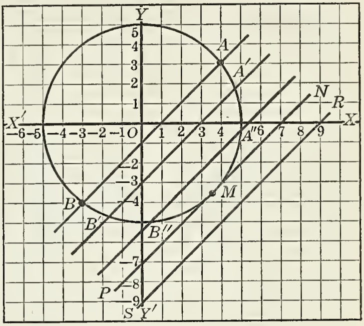
and y = number of feet in a side of the smaller square,

then

$$x^2 + y^2 = 25 \qquad (1)$$

$$x - y = 1. \qquad (2)$$

Graphic method of solving one linear and one quadratic equation. The graph of (1) is the circle shown in the adjoining figure; the graph of (2) is the straight line AB. The two points of intersection are A and B. The coördinates of A are $x = 4$, $y = 3$; of B, $x = -3$, $y = -4$.



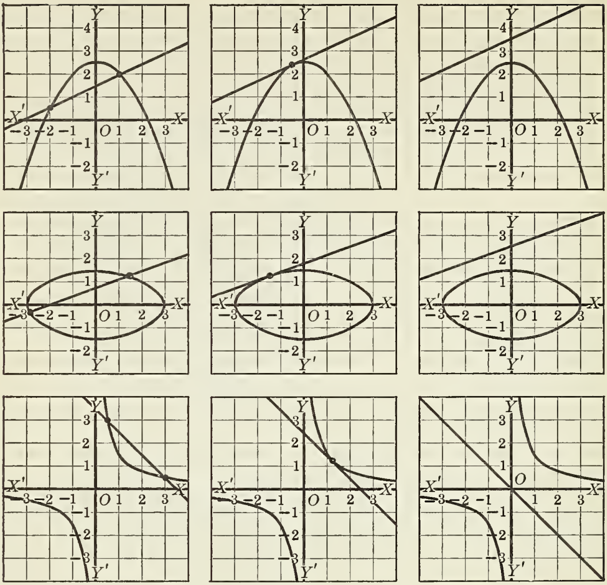
Notice that the second set of answers, $x = -3$, $y = -4$, has no meaning for this problem. Why? The answer therefore is 4 feet and 3 feet.

This problem illustrates the graphic treatment of a circle and straight line. Note that if the line AB were moved so that it remained parallel to its original position, the points A and B would get closer and closer together and finally coincide at M, at which point the two roots would be equal. The line NMP would then be tangent to the circle. If the line moved still farther, it would not intersect the circle and there could be no real roots. The roots in that case would be imaginary.

Note that real roots of simultaneous equations correspond to points of intersection, and imaginary roots indicate that the graphs do not intersect.

Also observe that since a straight line cannot intersect a circle in more than two points, the maximum number of solutions is two.

Since the graph of a quadratic equation may be a parabola, an ellipse, or an hyperbola, as well as a circle, we might have occasion to use any one of these other conic sections in conjunction with the straight line. The discussion as to the number of intersections is similar to that given for the circle, as illustrated in the diagram at the top of page 296.



Algebraic method of solving one linear and one quadratic equation. In solving simultaneous equations, elimination by substitution is generally the most convenient method. To solve the problem on page 294 by this method, solve the linear equation for one unknown in terms of the other, and substitute the result in the quadratic.

The equations of the problem are:

$$x^2 + y^2 = 25 \qquad (1)$$

$$x - y = 1 \qquad (2)$$

Solving (2) for x in terms of y,

$$x = y + 1. \qquad (3)$$

Substituting $(y + 1)$ for x in the quadratic equation (1), we have

$$(y + 1)^2 + y^2 = 25.$$
$$y^2 + 2y + 1 + y^2 = 25.$$
$$2y^2 + 2y - 24 = 0.$$
$$y^2 + y - 12 = 0.$$

Factoring $\quad (y + 4)(y - 3) = 0.$

$y + 4 = 0.$	$y - 3 = 0.$
$y = -4.$	$y = +3.$
Substituting in (3) $\quad x = -3.$	$x = 4.$

Check

$x = -3,\ y = -4.$	$x = 4,\ y = 3.$
In (1) $(-3)^2 + (-4)^2 = 25.$	In (1) $16 + 9 = 25.$
$25 = 25.$	$25 = 25.$
In (2) $(-3) - (-4) = 1.$	In (2) $4 - 3 = 1.$
$1 = 1.$	$1 = 1.$

As stated in the graphic method, the answer to this problem is 4 feet and 3 feet. Why can't $x = -3$, $y = -4$ be used?

EXERCISES

Group A

Solve each of the following (*a*) graphically and (*b*) algebraically. As a check, compare the answers of (*a*) and (*b*).

1. $x^2 + y^2 = 16$
$x = y.$

2. $x^2 + y^2 = 25$
$y = 2x + 5.$

3. $x^2 = y$
$x - y = \frac{1}{4}.$

4. $x^2 - y = 1$
$y = x + 1.$

5. $y = x^2 - 5$
$2x - y = 2.$

6. $xy = 6$
$2x - y = 4.$

7. $xy = 12$
$x - y = 1.$

8. $x^2 + y^2 = 100$
$3x - 2y = 12.$

Solve the following by the graphic method and then by the algebraic method.

9. Find two numbers whose sum is 6 and whose product is 8.

10. The sum of the radii of two circles is 7 in., and the sum of their areas is 25π sq. in. Find the radii.

11. The hypotenuse of a right triangle is 13 in. The sum of the two legs is 17 in. Find the lengths of the legs.

12. The area of a rectangle is 8 sq. in. and its perimeter is 12 in. Find its dimensions.

13. Find the dimensions of a rectangle such that its perimeter is 28 in. and its diagonal is 10 in.

Group B

14. $x^2 + 4y^2 = 16$
$y + x + 1 = 0.$

15. $4x^2 + 9y^2 = 36$
$y = x + 1.$

16. $x^2 - 4y^2 = 9$
$5y = x + 5.$

17. $y^2 - 2x^2 = 2$
$y = x + 1.$

18. $4x^2 + 4y^2 = 25$
$2x + 2y = 7.$

Group C

19. $x^2 - xy + y^2 = 7$
$x + y = 5.$

20. $x^2 + 3xy = 22$
$x + y = 5.$

21. $\dfrac{1}{x} - \dfrac{1}{y} = -\dfrac{1}{2}$
$x = 3y - 1.$

22. $x^2 - 2y = 10$
$3x - 2y = 6.$

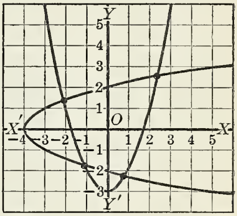
Two Quadratic Equations

Graphic method of solving two quadratic equations. The graph of a quadratic equation having two variables may be a circle, a parabola, an ellipse, or an hyperbola. From the intersections we may sometimes find all the roots, while at other times, imaginary roots will be indicated by the relative positions of the curves.

1. *Two parabolas.*

Let it be required to solve graphically the two equations:

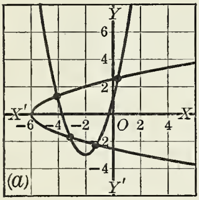
$$y^2 = x + 4$$
$$x^2 = y + 3.$$

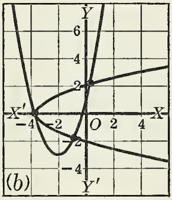


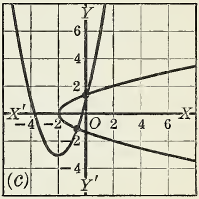
The graphs of these equations, as shown in the diagram, are two parabolas, intersecting in 4 points.

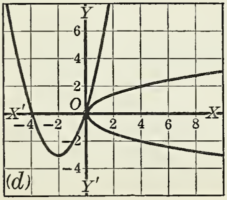
The coördinates of each intersection point give a solution of the two equations.

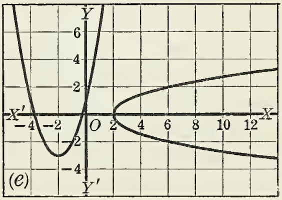
Note, however, the following positions that may arise and the effect upon the number of real and imaginary solutions.









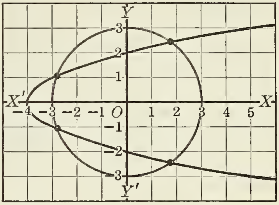


2. *Circle and Parabola.*

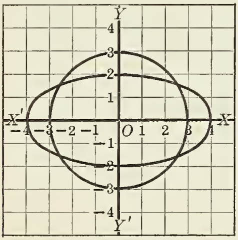
Solve graphically:

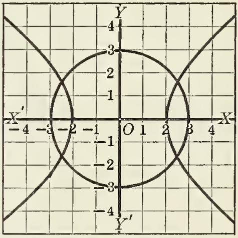
$$x^2 + y^2 = 9$$
$$y^2 = x + 4.$$

The graph of the first equation is a circle and of the second a parabola. As seen in the diagram, their intersections give 4 sets of roots.



The graphs below show two other possible combinations of the conic sections. Study each figure and then draw the graphs in other relative positions.





Algebraic method of solving simultaneous quadratic equations. Most frequently, simultaneous quadratic equations are solved by the methods of elimination you have learned to use in solving simultaneous linear equations. However, to facilitate the work, we frequently use special devices, and these can best be understood from the following illustrative examples.

Illustrative examples.

Example 1. Solve $2x^2 - y^2 = 23$ (1)

$x^2 + 2y^2 = 34.$ (2)

Solution

$$\begin{array}{r|l} 2 & 2x^2 - y^2 = 23 \\ 1 & x^2 + 2y^2 = 34 \end{array}$$

$$\begin{aligned} 4x^2 - 2y^2 &= 46 \\ x^2 + 2y^2 &= 34 \\ \hline 5x^2 &= 80 \\ x^2 &= 16. \\ x &= \pm 4. \end{aligned}$$

Substituting $+4$ for x in (2)

$$\begin{aligned} 16 + 2y^2 &= 34. \\ y^2 &= 9. \\ \therefore y &= \pm 3. \end{aligned}$$

Substituting -4 for x in (2)

$$\begin{aligned} 16 + 2y^2 &= 34. \\ y^2 &= 9. \\ y &= \pm 3. \end{aligned}$$

Arranging these roots in tabular form:

$x =$	$+4$	$+4$	-4	-4
$y =$	$+3$	-3	$+3$	-3

Check by substituting each set in both original equations.

Example 2. Solve $x^2 + y^2 = 29$ (1)

$xy = 10.$ (2)

Solution

Solving (2) for x in terms of y, $x = \frac{10}{y}$. (3)

Substituting this value of x in equation (1),

$$\begin{aligned} \left(\frac{10}{y}\right)^2 + y^2 &= 29. \\ \frac{100}{y^2} + y^2 &= 29. \\ 100 + y^4 &= 29y^2. \\ y^4 - 29y^2 + 100 &= 0. \\ (y^2 - 25)(y^2 - 4) &= 0. \end{aligned}$$

$$\begin{aligned} y^2 - 25 &= 0. \\ y^2 &= 25. \\ y &= \pm 5. \end{aligned} \qquad \begin{aligned} y^2 - 4 &= 0. \\ y^2 &= 4. \\ y &= \pm 2. \end{aligned}$$

Substituting in equation (3)

$$\begin{aligned} x &= \frac{10}{\pm 5}. \\ x &= 2, -2. \end{aligned} \qquad \begin{aligned} x &= \frac{10}{\pm 2}. \\ x &= 5, -5. \end{aligned}$$

Arranging the roots in tabular form, we have:

$x =$	2	-2	5	-5
$y =$	5	-5	2	-2

Check by substituting each set in both original equations.

Example 3. Solve $x^2 - xy + y^2 = 3$

$$x^2 + xy + y^2 = 7.$$

Analysis

The plan of solution is to so combine the two given equations that another equation of the form $ax^2 + bxy + cy^2 = 0$ is obtained which can be factored into two linear equations. This is accomplished by eliminating the constant terms.

Solution

Eliminate the constant terms.

$$\begin{array}{r|rcrcrcl} 7 & x^2 & - & xy & + & y^2 & = & 3 \\ 3 & x^2 & + & xy & + & y^2 & = & 7 \\ \hline & 7x^2 & - & 7xy & + & 7y^2 & = & 21 \\ & 3x^2 & + & 3xy & + & 3y^2 & = & 21 \\ \hline & 4x^2 & - & 10xy & + & 4y^2 & = & 0 \\ & 2x^2 & - & 5xy & + & 2y^2 & = & 0. \end{array}$$

Factoring $(2x - y)(x - 2y) = 0.$

Using each of these linear equations with the same one of the original quadratic equations, we have:

$$x^2 + xy + y^2 = 7. \qquad\qquad x^2 + xy + y^2 = 7.$$
$$2x - y = 0. \qquad\qquad x - 2y = 0.$$

Solving each pair as a set of simultaneous quadratic and linear equations, we have the roots:

$x =$	$+1$	-1	$+2$	-2
$y =$	$+2$	-2	$+1$	-1

Check by substituting each set in both of the original equations.

EXERCISES

Group A

Solve (*a*) graphically, (*b*) algebraically, each of the following systems of equations. As a check, compare the answers of (*a*) and (*b*).

1. $x^2 + y^2 = 5$
$x^2 - y^2 = 3.$

2. $x^2 + y^2 = 13$
$x^2 - y^2 = 5.$

3. $x^2 + y^2 = 4$
$y^2 = x - 2.$

4. $x^2 + y^2 = 17$
$xy = 4.$

5. $3x^2 + 2xy = 16$
$x^2 + 2xy = 8.$

6. $x^2 + y^2 = 10$
$xy = -3.$

Solve and check:

7. $5x^2 + 3y^2 = 83$
$3x^2 - 7y^2 = 41.$

8. $6x^2 - 5y^2 = 34$
$8x^2 - 4y^2 = 56.$

9. $2x^2 + y^2 = 19$
$x^2 + 2y^2 = 11.$

10. $x^2 - y^2 + 8 = 0$
$y^2 = 6x.$

11. $x^2 + y^2 + 9x + 14 = 0$
$y^2 = 16 + 4x.$

Group B

Solve and check:

12. $x^2 + y^2 = 26$
$x - 2y^2 = 3.$

13. $2x^2 = 5xy - 2$
$x^2 = xy + 2.$

14. $x^2 + xy = 88$
$x^2 - xy = 40.$

15. $x^2 + y^2 = 10$
$y^2 = 3x.$

16. $x^2 - y^2 = 4$
$xy = \sqrt{5}$

17. $x^2 - xy + 4y^2 = 4$
$xy = 1.$

Group C

18. $x^2 + xy + y^2 = 13$
$x^2 - xy + y^2 = 7.$

19. $x^2 + xy = 12$
$xy - y^2 = 2.$

20. $x^2 - xy - 2y^2 = 0$
$x^2 + y = 5.$

21. $x^2 + y^2 + x + y = 6$
$xy = -2.$

22. $x^2 - y^2 = 45$
$xy + y^2 = 18.$

23. $x^3 + y^3 = 28$
$x + y = 4.$

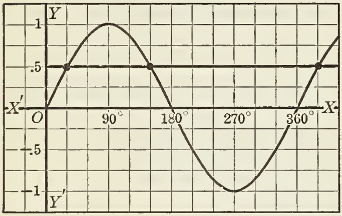
In each of the following, divide the first equation by the second:

24. $x^3 + y^3 = 28$
$x^2 - xy + y^2 = 7.$

25. $x^3 + y^3 = 243$
$xy(x + y) = 162.$

Systems of Trigonometric Equations

Illustrative example. Solve both graphically and algebraically



$$\sin x = y$$
$$y = \tfrac{1}{2}.$$

Solution

(*a*) Graphically

Assuming values for x from 0° to + 360° and computing the corresponding values for y in the equation $\sin x = y$, we have the adjoining diagram.

Now plotting the equation $y = \frac{1}{2}$, we have the line indicated on the diagram.

The abscissas of the points of intersection, 30° and 150°, are the solutions of the equations. Is 390° also a solution?

(*b*) Algebraically

Substitute $y = \frac{1}{2}$ in $\sin x = y$.

$$\sin x = \tfrac{1}{2}.$$
$$\therefore x = 30°, 150°, \text{etc.}$$

EXERCISES

Solve graphically and check algebraically:

1. $\sin x = y$
$y = 1.$

2. $\cos x = y$
$y = \frac{1}{2}.$

3. $\sin x = y$
$y = \frac{1}{3}.$

4. $\cos x = y$
$2y + 1 = 0.$

5. $\tan x = y$
$y = 2.$

6. $\sin x = y$
$y = \frac{1}{2}\sqrt{2}.$

7. $\tan x = y$
$y - 1 = 0.$

8. $\sin x = y$
$10y - 6 = 0.$

MISCELLANEOUS EXAMPLES

Solve the following systems of equations.

Group A

1. $xy = 5$
$2x + y = 7.$

2. $a^2 + b^2 = 25$
$a^2 + 2b^2 = 34.$

3. $x^2 + y^2 = 25$
$x^2 - 5 = y.$

4. $x^2 - xy + y^2 = 63$
$x = y - 3.$

5. $x^2 + xy = 12$
$xy - 2y^2 = 1.$

6. $\frac{x}{y} - \frac{y}{x} = \frac{5}{6}$
$x - y = 1.$

7. $xy + x = 9$
$xy + y = 8.$

8. $x^2 - xy + y^2 = 21$
$y^2 - 2xy + 15 = 0$

9. $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$
$x + y = 5.$

10. $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$
$xy = 18.$

Group B

11. $x^2 + y^2 = 13a^2$
$x - y = a.$

12. $3x^2 + y^2 = 2m^2$
$x^2 - 4y^2 = 5m^2.$

13. $x^2 + 3xy = -14$
$xy + 4y^2 = 30.$

14. $x^2 + y^2 = 5m^2$
$x + y = 3m.$

15. $x^3 + y^3 = 37$
$x^2 - xy + y^2 = 37.$

16. $x^3 + y^3 = 35$
$x + y = 5.$

Group C

17. $x^4 - y^4 = 65$
$x^2 - y^2 = 5.$

18. $x^4 - y^4 = 15$
$x^2 + y^2 = 5.$

19. $x^4 + x^2y^2 + y^4 = 21$
$x^2 + xy + y^2 = 7.$

20. $\frac{1}{x} - \frac{1}{y} = 1$
$\frac{1}{x^2} + \frac{1}{y^2} = 13.$

21. $2x^2 + 3xy = 9$
$3y^2 + 2xy = -3.$

PROBLEMS INVOLVING SIMULTANEOUS QUADRATIC EQUATIONS

1. The product of two numbers is 4 and the sum of their squares is 17. Find the numbers.

2. Two numbers are in the ratio of 3 : 4, and the difference of their squares is 63. Find the numbers.

3. Two cubical boxes are placed so that the smaller stands upon the larger. If their combined height is 4 feet and their combined volume is 28 cubic feet, find the length of an edge of each cube.

4. The hypotenuse of a right triangle is 25, and the sum of the two legs is 31. Find the length of each leg.

5. A piece of wire 60 inches long is bent into the form of a right triangle whose hypotenuse is 25 inches. Find the other two sides of the triangle.

6. The sum of the radii of two circles is 7 inches. The sum of the areas of these two circles equals the area of a circle whose radius is 5 inches. Find the radii.

7. Find two numbers whose sum multiplied by the greater is 24, and whose difference multiplied by the smaller is 4.

8. The sum of the reciprocals of two numbers is 7 and the sum of the reciprocals of their squares is 25. Find the numbers.

9. A sum of money at simple interest amounts in three years to \$460. Had the principal been $\frac{1}{4}$ greater and the rate 1% higher it would have amounted to \$590. Find the principal and the rate.

Discovering the Unknown

Although equations involving more than one unknown quantity were known in Egypt, Greece, and India, yet Diophantus (275 A.D.) was the first to give these quantities the special names "the first number," "the second number," etc. In general, however, his next step was to express each quantity in terms of a single quantity.

This method was used until about 1559, when Buteo introduced the letters A, B, C, etc., for the different unknowns, and the procedure of "addition and subtraction" to solve simultaneous equations.

The latter system prevailed until 1637, when Descartes used x, y, and z for the unknown quantities, as we do today.

CUMULATIVE REVIEW

Chapters VII, VIII, IX

1. Which of these statements are true? Which are false?

(*a*) The length of the line joining the points $(-3, 7)$ and $(-3, -6)$ is 1.

(*b*) If θ is a negative angle, $\sin^2\theta + \cos^2\theta = 1$.

(*c*) To find the y-intercept of the graph of a straight line, substitute zero for y.

(*d*) The graph of $y - 5 = 0$ is a straight line.

(*e*) The graph of the equation $y = 2x - 2$ has a slope equal to its y-intercept.

(*f*) $x = 1$ and $y = 5$ are the only values which will satisfy both equations: $x + y = 6$ and $y = 5x$.

(*g*) If $x : y = 3 : 4$, then $\frac{x^2}{y^2} = \frac{9}{16}$.

(*h*) The point $(4, 2)$ lies on the graph of $y = x + 2$.

(*i*) From the equations: $x + y + z = 25$ and $x - y - z = 19$, only one common set of values of x, y, and z can be found.

(*j*) The equation of the parabola passing through three points can be found, if the coördinates of the three points are given.

2. Complete each of the following statements:

(*a*) If $\tan x = -\frac{3}{4}$ and x is in quadrant II, the value of $\sec x$ is ? than 1.

(*b*) The value of $\sin\left(-\frac{3\pi}{4}\right)$ is ? .

(*c*) The cost of a railroad ticket varies with the distance traveled. Hence the ? is a function of the ? .

(*d*) If the discriminant of a quadratic equation is greater than zero, the roots of the equation must be ? and ? .

(*e*) In the equation $x^2 - 2x + 1 = 0$, the sum of the roots exceeds the product of the roots by ? .

(*f*) If $x = \frac{5}{2y} - 1$, then $(x + 1)^2 =$? .

(*g*) If $x = z$ when $y = 2x$ and $x + 2y + 3z = 30$, then $y =$? and $z =$? .

(*h*) A certain number of three digits is represented by $100h + 90 + u$. If the digits of this number be reversed, the new number, written in terms of h and u, is ? .

(*i*) If a boat sails upstream at the rate of 12 miles an hour in a current that flows at the rate of 2 miles an hour, the rate of the boat down the same stream would be ? miles an hour.

(*j*) The graph of $2x^2 + 2y^2 = 50$ is a ? .

3. If x is an acute angle, show geometrically that the functions of $180° + x$ are equal in absolute value to the same named functions of x.

4. At a certain point the length of a pond subtends an angle of 97°. If the distances from this point to the two ends of the pond are 100 feet and 80 feet respectively, find the length of the pond.

5. Write the equation of the line whose slope is -4 and whose y-intercept is $\frac{9}{5}$.

6. (*a*) Draw the graph of $y = x^2 - 2x - 1$ from $x = -2$ to $x = 4$ inclusive.

(*b*) Write the coördinates of the turning point of this graph.

(*c*) Does the turning point represent a maximum or a minimum value of y?

(*d*) Draw the axis of symmetry and write its equation.

(*e*) From the graph made in answer to (*a*), estimate correct to the nearest tenth the roots of $x^2 - 2x - 1 = 0$.

7. The cost of manufacturing an article varies directly as the wage paid to the worker. If 144 articles can be manufactured for $7 in wages, what will be the amount in wages for manufacturing 960 articles?

Solve for x and y:

8. $\dfrac{1}{x} + \dfrac{3}{y} = 8$

$\dfrac{3}{x} + \dfrac{2}{y} = 3.$

9. $y = x \tan 45°$

$y = (x - 300) \tan 60°.$

10. Find the equation of the straight line passing through the points (4, 9) and (-2, 3).

11. The sum of the digits of a two-digit number is 9. If the digits are reversed, the new number will be 9 less than twice the original number. Find the original number.

12. Solve graphically and check algebraically:

$$x^2 + 2y = 8$$
$$x - y = 0.$$

CHAPTER X. RADICALS

Mathematics is the only science which can develop a knowledge of absolute truth. — George B. Olds.

Radicals

In your work in geometry, physics, mechanics, and other sciences you will frequently find it necessary to deal with radical signs. In order to manipulate such expressions with facility and to form a background for later studies, we shall review briefly the work you have had in elementary algebra.

Recall fact 49. *In the expression* $\sqrt[4]{81} = 3$, 81 *is the radicand,* 3 *is the root, and* 4 *is the root index (ordinarily no root index is used in square root).*

Recall fact 50. *The principal root of a radical expression is its one real root if it has but one, or its positive real root if it has two real roots;* e.g., $\sqrt{9} = \pm 3$, *principal root is* $+3$; $\sqrt[3]{8} = 2$ *(principal root)*; $\sqrt[3]{-8} = -2$ *(principal root).*

Throughout this book, unless otherwise indicated, the principal root only will be used.

Recall fact 51. *Fundamental law of radicals:*

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$$

$$\text{or} \quad \sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

Simplification of Radicals

It is frequently desirable to change the form of a radical so that its numerical value may be computed more easily. In transforming a radical recall that the quantity under the radical sign: (*a*) must not be in the form of a fraction; (*b*) must contain no factors which are powers of the root index; and (*c*) the order (*i.e.*, root index) of the radical must be as low as possible.

Illustrative examples.

Example 1. $\sqrt{28} = \sqrt{4 \times 7} = 2\sqrt{7}.$

Example 2. $\sqrt{27\,a^3} = \sqrt{9\,a^2 \times 3\,a} = 3\,a\sqrt{3\,a}.$

Example 3. $\sqrt[3]{54} = \sqrt[3]{27 \times 2} = 3\sqrt[3]{2}.$

Example 4. $\sqrt[3]{16\,a^4b^3} = \sqrt[3]{8\,a^3b^3 \times 2\,a} = 2\,ab\sqrt[3]{2\,a}.$

Example 5. $\sqrt{\frac{1}{2}} = \sqrt{\frac{1}{2} \times \frac{2}{2}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{4} \times 2} = \frac{1}{2}\sqrt{2}.$

Example 6. $\sqrt{\frac{3}{8}} = \sqrt{\frac{3}{8} \times \frac{2}{2}} = \sqrt{\frac{6}{16}} = \sqrt{\frac{1}{16} \times 6} = \frac{1}{4}\sqrt{6}.$

Example 7. $\frac{2}{3}\sqrt{2\frac{2}{3}} = \frac{2}{3}\sqrt{\frac{8}{3} \times \frac{3}{3}} = \frac{2}{3}\sqrt{\frac{24}{9}} = \frac{2}{3}\sqrt{\frac{4}{9} \times 6}$

$$= \left(\frac{2}{3} \times \frac{2}{3}\right)\sqrt{6} = \frac{4}{9}\sqrt{6}.$$

Example 8. $a\sqrt{\frac{b}{a}} = a\sqrt{\frac{b}{a} \times \frac{a}{a}} = a\sqrt{\frac{ab}{a^2}} = a\sqrt{\frac{1}{a^2} \times ab}$

$$= \left(a \times \frac{1}{a}\right)\sqrt{ab} = \sqrt{ab}.$$

Example 9. $\sqrt[3]{\frac{b}{3\,a}} = \sqrt[3]{\frac{b}{3\,a} \times \frac{9\,a^2}{9\,a^2}} = \sqrt[3]{\frac{9\,a^2b}{27\,a^3}}$

$$= \sqrt[3]{\frac{1}{27\,a^3} \times 9\,a^2b} = \frac{1}{3\,a}\sqrt[3]{9\,a^2b}.$$

Example 10. $\sqrt{-18} = \sqrt{9 \times (-2)} = 3\sqrt{-2}.$

Example 11. $\sqrt[3]{-54} = \sqrt[3]{-27 \times 2} = \sqrt[3]{(-3)^3 \times 2} = -3\sqrt[3]{2}.$

Observe that if the radicand contains a fraction and the root index indicates a square root, the denominator is made a perfect square; if the root index indicates a cube root, the denominator is made a perfect cube.

EXERCISES

Simplify each of the following:

1. $\sqrt{8}$.
2. $\sqrt{18}$.
3. $\sqrt{12}$.
4. $\sqrt{27}$.
5. $2\sqrt{32}$.
6. $3\sqrt{50}$.
7. $\frac{2}{5}\sqrt{75}$.
8. $\sqrt[3]{-16}$.
9. $\sqrt[3]{54}$.
10. $\sqrt[3]{128}$.
11. $3\sqrt[3]{81}$.
12. $\frac{2}{3}\sqrt[3]{40}$.
13. $\frac{1}{2}\sqrt[3]{108}$.
14. $\sqrt{\frac{5}{9}}$.
15. $\sqrt{\frac{5}{4}}$.
16. $\sqrt{\frac{5}{16}}$.
17. $\sqrt{\frac{3}{8}}$.
18. $\sqrt{\frac{2}{3}}$.
19. $\frac{2}{3}\sqrt{\frac{3}{5}}$.
20. $\frac{1}{2}\sqrt{\frac{7}{12}}$.
21. $\frac{3}{4}\sqrt{\frac{9}{32}}$.
22. $\sqrt{\frac{5}{27}}$.
23. $\sqrt[3]{\frac{1}{2}}$.
24. $\sqrt[3]{\frac{1}{3}}$.
25. $\frac{1}{2}\sqrt[3]{\frac{3}{4}}$.
26. $\sqrt[3]{1\frac{1}{2}}$.
27. $\sqrt[3]{1\frac{1}{3}}$.
28. $\sqrt{x^3}$.
29. $\sqrt{x^3y^5}$.
30. $\sqrt{9\,m^2y^3}$.
31. $3\sqrt{27\,a^3b^4}$.
32. $\frac{3}{2}\sqrt{18\,xy^5}$.
33. $\sqrt[3]{x^4y^4}$.
34. $\sqrt[3]{ab^5}$.
35. $4\sqrt[3]{27\,m^8}$.
36. $5\sqrt[3]{54\,a^6b}$.
37. $\frac{5}{2}\sqrt[3]{16\,a^4b^3}$.
38. $\sqrt{\frac{1}{x}}$.
39. $\sqrt{\frac{1}{y}}$.
40. $\sqrt{\frac{1}{x^3}}$.
41. $3\sqrt{\frac{2}{x^5}}$.
42. $4\sqrt{\frac{m^2}{x^7}}$.
43. $\frac{b}{a}\sqrt{\frac{a}{bc}}$.
44. $\frac{1}{a}\sqrt{\frac{c^3}{a^2b}}$.
45. $\frac{b}{a}\sqrt{\frac{x}{8\,ab^3}}$.
46. $\frac{3}{2}\sqrt{\frac{x^5}{2\,a^2b^3c}}$.
47. $\sqrt[3]{\frac{1}{x^2}}$.
48. $\sqrt[3]{\frac{1}{y^4}}$.
49. $\sqrt[3]{\frac{x}{y}}$.
50. $\sqrt[3]{\frac{x^2}{2\,y}}$.
51. $4\sqrt[3]{\frac{x^2}{y^7}}$.
52. $\frac{m}{n}\sqrt[3]{\frac{y}{mn}}$.
53. $\frac{a}{b}\sqrt{\frac{y^3}{a^2b}}$.
54. $\sqrt[4]{32}$.
55. $\sqrt[4]{243}$.
56. $\sqrt[5]{64}$.

FUNDAMENTAL OPERATIONS WITH RADICALS

Addition and subtraction of radicals.

Recall fact 52. *Similar radicals are radicals which have the same root index and the same radicand.*

Recall fact 53. *Similar radicals may be added and subtracted.*

Illustrative examples.

Example 1. Simplify $5\sqrt{3}+3\sqrt{5}-\sqrt{3}-4\sqrt{5}$.

Solution

$$\begin{aligned}5\sqrt{3}+3\sqrt{5}-\sqrt{3}-4\sqrt{5} &= (5-1)\sqrt{3}+(3-4)\sqrt{5}\\ &= 4\sqrt{3}-\sqrt{5}.\end{aligned}$$

Example 2. Simplify $2\sqrt{\frac{5}{3}}-\sqrt{60}-5\sqrt{\frac{3}{5}}$.

Solution

$$\begin{aligned}2\sqrt{\tfrac{5}{3}}-\sqrt{60}-5\sqrt{\tfrac{3}{5}} &= 2\sqrt{\tfrac{5}{3}\times\tfrac{3}{3}}-\sqrt{4\times 15}-5\sqrt{\tfrac{3}{5}\times\tfrac{5}{5}}\\ &= 2\sqrt{\tfrac{15}{9}}-2\sqrt{15}-5\sqrt{\tfrac{15}{25}}\\ &= 2\sqrt{\tfrac{1}{9}\times 15}-2\sqrt{15}-5\sqrt{\tfrac{1}{25}\times 15}\\ &= \tfrac{2}{3}\sqrt{15}-2\sqrt{15}-\sqrt{15}\\ &= (\tfrac{2}{3}-2-1)\sqrt{15}\\ &= -2\tfrac{1}{3}\sqrt{15} \text{ or } -\tfrac{7}{3}\sqrt{15}.\end{aligned}$$

Example 3. Simplify $4\sqrt[3]{\frac{a^2x}{4}}-3\sqrt[3]{\frac{x}{32\,a}}-\frac{2}{a}\sqrt[3]{2\,a^5x}$.

Solution

$$\begin{aligned}&4\sqrt[3]{\frac{a^2x}{4}}-3\sqrt[3]{\frac{x}{32\,a}}-\frac{2}{a}\sqrt[3]{2\,a^5x}\\ &\quad= 4\sqrt[3]{\frac{a^2x}{4}\times\frac{2}{2}}-3\sqrt[3]{\frac{x}{32\,a}\times\frac{2\,a^2}{2\,a^2}}-\frac{2}{a}\sqrt[3]{a^3\times 2\,a^2x}\\ &\quad= 4\sqrt[3]{\frac{2\,a^2x}{8}}-3\sqrt[3]{\frac{2\,a^2x}{64\,a^3}}-2\sqrt[3]{2\,a^2x}\\ &\quad= 4\sqrt[3]{\frac{1}{8}\times 2\,a^2x}-3\sqrt[3]{\frac{1}{64\,a^3}\times 2\,a^2x}-2\sqrt[3]{2\,a^2x}\\ &\quad= 2\sqrt[3]{2\,a^2x}-\frac{3}{4\,a}\sqrt[3]{2\,a^2x}-2\sqrt[3]{2\,a^2x}\\ &\quad= \left(2-\frac{3}{4\,a}-2\right)\sqrt[3]{2\,a^2x}=-\frac{3}{4\,a}\sqrt[3]{2\,a^2x}.\end{aligned}$$

EXERCISES

Simplify:

1. $\sqrt{18}+\sqrt{50}-\sqrt{72}$.
2. $\sqrt{12}+\sqrt{27}-\sqrt{8}+\sqrt{18}$.
3. $\sqrt{48}+2\sqrt{75}-3\sqrt{12}$.
4. $\sqrt{75}-4\sqrt{243}+2\sqrt{108}$.
5. $6\sqrt{24}+2\sqrt{150}-4\sqrt{54}$.
6. $\sqrt{20}+\sqrt{125}-2\sqrt{180}$.
7. $2\sqrt{3}-\frac{1}{2}\sqrt{12}+4\sqrt{27}$.
8. $\sqrt[3]{81}+\sqrt[3]{3}-\sqrt[3]{24}$.
9. $2\sqrt[3]{2}-4\sqrt[3]{16}-3\sqrt[3]{54}$.
10. $2\sqrt[3]{32}+3\sqrt[3]{108}+\sqrt[3]{500}$.
11. $3\sqrt[3]{54}+4\sqrt[3]{16}-3\sqrt[3]{250}$.
12. $\sqrt{\frac{1}{7}}+\sqrt{63}+5\sqrt{7}$.
13. $2\sqrt{\frac{1}{2}}-4\sqrt{\frac{1}{8}}+8\sqrt{2}$.
14. $\sqrt{1\frac{1}{8}}+\sqrt{\frac{1}{2}}$.
15. $6\sqrt{\frac{2}{3}}-5\sqrt{24}+12\sqrt{\frac{3}{2}}$.
16. $\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}$.
17. $2\sqrt{1\frac{2}{3}}-3\sqrt{\frac{3}{5}}+\sqrt{\frac{4}{15}}$.
18. $\sqrt{9\,a}+\sqrt{25\,a}-\sqrt{64\,a}$.
19. $\sqrt{18\,y}-\sqrt{\frac{1}{2}\,y}+\sqrt{\frac{1}{8}\,y}$.
20. $\sqrt{98a^2}+2\sqrt{32a^2}-\sqrt{75a^2}$.
21. $\sqrt[3]{\frac{1}{4}}-\sqrt[3]{\frac{2}{27}}$.
22. $2\sqrt[3]{\frac{1}{2}}+4\sqrt[3]{\frac{1}{16}}$.
23. $\sqrt[3]{\frac{x^2}{9}}+3\sqrt[3]{\frac{3}{x}}$.
24. $a\sqrt[3]{\frac{x}{a^2}}+x\sqrt[3]{\frac{a}{x^2}}$.
25. $(x+1)\sqrt[3]{16\,a^2}+x\sqrt[3]{\frac{a^2}{4}}$.

Multiplication of Radicals. By means of the fundamental principle $\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$, radicals of the same order (root index) can be multiplied. However, be sure to express the result in simplest form.

Illustrative examples.

Example 1. $\sqrt{2}\times\sqrt{6}=\sqrt{12}=\sqrt{4\times 3}=2\sqrt{3}$.

Example 2. $3\sqrt{5}\times 2\sqrt{2}=6\sqrt{10}$.

Example 3. $\sqrt[3]{2}\times\sqrt[3]{16}=\sqrt[3]{32}=\sqrt[3]{8\times 4}=2\sqrt[3]{4}$.

Example 4. $\sqrt{3}\times\sqrt{3}=3$.

Example 5. $2\sqrt{5}\times 3\sqrt{5}=6\times 5=30$.

Example 6. $\sqrt{\frac{5}{2}}\times\sqrt{\frac{8}{5}}=\sqrt{\frac{5}{2}\times\frac{8}{5}}=\sqrt{4}=2$.

EXERCISES

Perform the indicated multiplication and express the result in its simplest form:

1. $\sqrt{2} \times \sqrt{8}$.
2. $\sqrt{50} \times \sqrt{2}$.
3. $\sqrt{30} \times \sqrt{\frac{3}{10}}$.
4. $\sqrt{\frac{2}{3}} \times \sqrt{\frac{27}{2}}$.
5. $\sqrt{\frac{1}{4}} \times \sqrt{\frac{4}{25}}$.
6. $2\sqrt{5} \times 3\sqrt{20}$.
7. $4\sqrt{18} \times \sqrt{2}$.
8. $\sqrt{8} \times \sqrt{3}$.
9. $\sqrt{27} \times \sqrt{2}$.
10. $4\sqrt{2} \times 2\sqrt{6}$.
11. $5\sqrt{5} \times 2\sqrt{10}$.
12. $3\sqrt{2} \times \sqrt{10}$.
13. $\sqrt{15} \times \sqrt{15}$.
14. $2\sqrt{2} \times 3\sqrt{2}$.
15. $4\sqrt{3} \times \sqrt{3}$.
16. $6\sqrt{5} \times 2\sqrt{5}$.
17. $2\sqrt{7} \times 3\sqrt{7}$.
18. $(2\sqrt{3})^2$.
19. $(-3\sqrt{2})^2$.
20. $(-5\sqrt{10})^2$.
21. $\sqrt[3]{2} \times \sqrt[3]{4}$.
22. $\sqrt[3]{4} \times \sqrt[3]{16}$.
23. $\sqrt[3]{8} \times \sqrt[3]{8}$.
24. $\sqrt[3]{\frac{2}{9}} \times \sqrt[3]{36}$.
25. $\sqrt[3]{\frac{2}{3}} \times \sqrt[3]{\frac{81}{16}}$.
26. $3\sqrt[3]{2} \times 2\sqrt[3]{4}$.
27. $\sqrt[3]{2} \times \sqrt[3]{16}$.
28. $2\sqrt[3]{4} \times 3\sqrt[3]{5}$.
29. $3\sqrt[3]{3} \times \sqrt[3]{18}$.
30. $\sqrt{a} \times \sqrt{4a}$.
31. $\sqrt{2xy} \times \sqrt{8xy}$.
32. $\sqrt{5x^2} \times \sqrt{10x^2}$.
33. $\sqrt{\frac{1}{a}} \times \sqrt{a^3}$.
34. $\sqrt{\frac{32x^2}{y^4}} \times \sqrt{\frac{y^6}{2}}$.
35. $2\sqrt[3]{2a^2} \times 4\sqrt[3]{4a}$.
36. $4\sqrt[3]{4} \times 3\sqrt[3]{16a^3}$.
37. $(2\sqrt[3]{7x^2})^3$.
38. $(2\sqrt[3]{4a})^3$.
39. $\sqrt[3]{2a^3} \times \sqrt[3]{4b}$.
40. $3\sqrt[3]{xy} \times \sqrt[3]{xy^5z}$.
41. $(3\sqrt[3]{5a^3})^3$.
42. $\sqrt[3]{ab} \times \sqrt[3]{b^2}$.
43. $\sqrt[3]{ab^2} \times \sqrt[3]{bc}$.
44. $\sqrt[3]{abc} \times \sqrt[3]{b^2c^5}$.

The form for the multiplication of polynomials involving radicals is similar to that used in the multiplication of algebraic polynomials. Be sure, however, to express the answers in simplest form.

Illustrative example.

Multiply $(3\sqrt{2} + 5\sqrt{3})$ by $(2\sqrt{2} - \sqrt{3})$.

Solution

$$\begin{array}{lllll}
3\sqrt{2} & + & 5\sqrt{3} & & \\
2\sqrt{2} & - & \sqrt{3} & & \\
\hline
6 \times 2 & + & 10\sqrt{6} & & \\
 & - & 3\sqrt{6} & - & 5 \times 3 \\
\hline
12 & + & 7\sqrt{6} & - & 15 \quad = 7\sqrt{6} - 3.
\end{array}$$

EXERCISES

Find the product of each of the following:

1. $(\sqrt{2} + \sqrt{3} - \sqrt{5})\sqrt{2}$.
2. $(4\sqrt{27} - \sqrt{75} - \sqrt{3})\, 3\sqrt{3}$.
3. $(2\sqrt{5} - 3\sqrt{125} - 2\sqrt{10})\, 2\sqrt{5}$.
4. $(\sqrt{18\,a} + \sqrt{50\,a} - 2\sqrt{8\,a})\sqrt{a}$.
5. $(3 + \sqrt{2})(3 - \sqrt{2})$.
6. $(\sqrt{5} + 2)(\sqrt{5} + 2)$.
7. $(\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5})$.
8. $(2\sqrt{3} - \sqrt{2})(3\sqrt{3} + \sqrt{2})$.
9. $(2\sqrt{5} + 3\sqrt{3})(2\sqrt{5} - \sqrt{3})$.
10. $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$.
11. $(\sqrt{2\,a} - \sqrt{b})(\sqrt{a} - \sqrt{2\,b})$.
12. $(3\sqrt{a} - 2\sqrt{2\,b})(3\sqrt{a} + \sqrt{2\,b})$.
13. $(\sqrt{3} - \sqrt{2})^2$.
14. $(2\sqrt{5} + 3\sqrt{2})^2$.
15. $(5\sqrt{3} + \sqrt{2})^2$.
16. $(a + \sqrt{b})^2$.
17. $(\sqrt{2\,x} + \sqrt{3\,a})^2$.
18. $(3\sqrt{x} - 2\sqrt{y})^2$.
19. $(\sqrt{3} + \sqrt{2} - \sqrt{5})(\sqrt{2} - \sqrt{3})$.
20. $(3\sqrt{2} - 2\sqrt{5} - 8\sqrt{3})(2\sqrt{2} + \sqrt{5} - \sqrt{6})$.
21. $(a\sqrt{b} - \sqrt{ab} + 3\sqrt{a})(\sqrt{a} + \sqrt{b})$.
22. $(\sqrt{a} - \sqrt{b} - \sqrt{x})(\sqrt{b} - \sqrt{x} + \sqrt{a})$.

23. If $x = \sqrt{5} - 1$, find the value of $x^2 + 2x - 4$.

24. If $y = \sqrt{3} + \sqrt{2}$, find the value of $y^2 - 2y - 7$.

25. If $x = \sqrt{2} - 1$ and $y = 1 + \sqrt{3}$, find the value of $x^2 + xy - 2$.

26. If $x = \dfrac{\sqrt{5} - 1}{2}$, find the value of $x^2 - 3x + 2$.

27. Prove that $2 + \sqrt{3}$ is a root of $x^2 - 4x + 1 = 0$.

28. Prove that $3 - \sqrt{5}$ is a root of $x^2 - 6x + 4 = 0$.

29. Prove that $2 - \sqrt{6}$ is a root of $x^2 - 4x - 2 = 0$.

30. Prove that $-3 + \sqrt{7}$ is a root of $x^2 + 6x + 2 = 0$.

Formation of quadratic equations when roots are irrational. The relation between the roots and coefficients of a quadratic equation may be used to form the equation whose roots are irrational.

Illustrative example. Form the equation whose roots are $3 + \sqrt{2}$ and $3 - \sqrt{2}$.

Solution

Here $\qquad r_1 = 3 + \sqrt{2}$ and $r_2 = 3 - \sqrt{2}$.

Using the formula $\qquad x^2 - (r_1 + r_2)x + r_1r_2 = 0$

and substituting, we have

$$x^2 - (3 + \sqrt{2} + 3 - \sqrt{2})x + (3 + \sqrt{2})(3 - \sqrt{2}) = 0.$$

$$\therefore x^2 - 6x + 7 = 0.$$

EXERCISES

Form the equation whose roots are:

1. $\sqrt{2} + 1, \sqrt{2} - 1$.
2. $\sqrt{3} + 2, \sqrt{3} - 2$.
3. $4 - \sqrt{2}, 4 + \sqrt{2}$.
4. $\sqrt{a}, -\sqrt{a}$.
5. $b + \sqrt{a}, b - \sqrt{a}$.
6. $\sqrt{3} + \sqrt{2}, \sqrt{3} - \sqrt{2}$.
7. $\sqrt{5} - \sqrt{3}, \sqrt{5} + \sqrt{3}$.
8. $\dfrac{a + \sqrt{b}}{2}, \dfrac{a - \sqrt{b}}{2}$.

Division of radicals. The primary purpose in simplifying radicals is to lessen the work of computation. For instance, to find the numerical value of $\sqrt{6} \div \sqrt{2}$, it is necessary to find the square roots of 6 and 2 respectively and then perform a tedious long division. If, however, we first simplify the expression $\sqrt{6} \div \sqrt{2}$ and obtain $\sqrt{3}$, it is then necessary to find the square root of 3 only.

Method 1. The division of radicals may be performed by the fundamental principle $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$; that is, radicands having the same root index may be incorporated under one radical.

Illustrative examples.

Example 1. $\dfrac{8\sqrt{6}}{12\sqrt{2}} = \dfrac{2}{3}\sqrt{\dfrac{6}{2}} = \dfrac{2}{3}\sqrt{3}.$

Example 2. $\dfrac{\sqrt{7}}{\sqrt{2}} = \sqrt{\dfrac{7}{2}} = \sqrt{\dfrac{7}{2} \times \dfrac{2}{2}} = \sqrt{\dfrac{14}{4}} = \sqrt{\dfrac{1}{4} \times 14} = \dfrac{1}{2}\sqrt{14}.$

EXERCISES

Simplify:

1. $\dfrac{\sqrt{12}}{\sqrt{3}}$.
2. $\dfrac{3\sqrt{6}}{\sqrt{3}}$.
3. $\dfrac{4\sqrt{18}}{\sqrt{3}}$.
4. $\dfrac{8\sqrt{12}}{2\sqrt{2}}$.
5. $\dfrac{8\sqrt{10}}{12\sqrt{5}}$.
6. $\dfrac{\sqrt{6}}{\sqrt{5}}$.
7. $\dfrac{\sqrt{2}}{\sqrt{3}}$.
8. $\dfrac{\sqrt{5}}{\sqrt{7}}$.
9. $\dfrac{4\sqrt{3}}{8\sqrt{8}}$.
10. $\dfrac{10\sqrt{8}}{12\sqrt{12}}$.
11. $\dfrac{\sqrt[3]{2}}{\sqrt[3]{16}}$.
12. $\dfrac{\sqrt[3]{7}}{\sqrt[3]{16}}$.

Method 2. To find the value of $\dfrac{3}{\sqrt{3}}$, where the denominator is a monomial radical, it is necessary to find the square root of 3 and then perform a long division. If we could transform this expression so that the denominator would be a rational number, the computation would be much easier.

To make the denominator of $\frac{3}{\sqrt{3}}$ rational, we will multiply both numerator and denominator by $\sqrt{3}$; thus,

$$\frac{3}{\sqrt{3}} = \frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3} = 1.73.$$

Making the denominator of a fraction into a rational number is called *rationalizing the denominator.*

To rationalize the denominator of a fraction, multiply the numerator and denominator by that quantity which will make the denominator rational.

Illustrative examples.

Example 1. Simplify $\sqrt{2} \div \sqrt{3}$.

Solution

$$\frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}.$$

Example 2. Simplify $\frac{\sqrt[3]{6}}{\sqrt[3]{4}}$.

Solution

$$\frac{\sqrt[3]{6}}{\sqrt[3]{4}} = \frac{\sqrt[3]{6}}{\sqrt[3]{4}} \times \frac{\sqrt[3]{2}}{\sqrt[3]{2}} = \frac{\sqrt[3]{12}}{2}.$$

Example 3. Find the value of $\frac{9}{\sqrt{3}}$ correct to the nearest tenth.

Solution

$$\frac{9}{\sqrt{3}} = \frac{9}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{9\sqrt{3}}{3} = 3\sqrt{3} = 3 \times 1.73$$
$$= 5.19 = 5.2.$$

EXERCISES

Rationalize the denominator and simplify:

1. $\frac{1}{\sqrt{2}}$.
2. $\frac{2}{\sqrt{3}}$.
3. $\frac{7}{\sqrt{7}}$.
4. $\frac{3}{\sqrt{8}}$.

5. $\dfrac{6}{\sqrt{32}}$.
6. $\dfrac{8}{2\sqrt{3}}$.
7. $\dfrac{2}{3\sqrt{8}}$.
8. $\dfrac{\sqrt{5}}{\sqrt{3}}$.
9. $\dfrac{\sqrt{3}}{\sqrt{2}}$.
10. $\dfrac{12\sqrt{2}}{2\sqrt{3}}$.
11. $\dfrac{3\sqrt{7}}{2\sqrt{6}}$.
12. $\dfrac{1}{\sqrt[3]{2}}$.
13. $\dfrac{3}{\sqrt[3]{5}}$.
14. $\dfrac{3}{\sqrt[3]{9}}$.
15. $\dfrac{a}{\sqrt{a}}$.
16. $\dfrac{a}{\sqrt{ab^2}}$.
17. $\dfrac{\sqrt{a}}{\sqrt{b}}$.
18. $\dfrac{a}{\sqrt[3]{a}}$.
19. $\dfrac{b}{\sqrt[3]{a^2}}$.
20. $\dfrac{xy}{\sqrt[3]{x^2y}}$.

The following exercises will help you to learn how to rationalize a denominator that is a binomial radical expression.

Complete the following:

1. $(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2}) = ?$
2. $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2}) = ?$
3. $(\sqrt{3}-\sqrt{5})(\sqrt{3}+\sqrt{5}) = ?$
4. $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = ?$
5. $(\sqrt{5}+\sqrt{3})(\ ?\ ) = 5 - 3 = 2.$
6. $(\sqrt{8}-\sqrt{2})(\ ?\ ) = 8 - 2 = 6.$
7. $(\sqrt{10}-\sqrt{3})(\ ?\ ) = 10 - 3 = 7.$
8. $(\sqrt{2}+3)(\ ?\ ) = 2 - 9 = -7.$

You will observe in the examples above that a binomial involving radicals can be rationalized by multiplying it by another binomial alike in all respects but having the opposite connecting sign. Binomials having such a relation are called *conjugate binomials.*

Illustrative example. Find the value of $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ correct to the nearest tenth.

Without a knowledge of radicals, to find the value of $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ correct to the nearest tenth would involve finding the square roots of 3 and 2, and then dividing their difference by their sum, a rather long and tedious procedure. However, by rationalizing the denominator it is easily seen that the work is considerably lessened.

Solution

$$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}\cdot\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}=\frac{5-2\sqrt{6}}{1}$$

$$=\frac{5-2(2.44)}{1}=5-4.88=.12$$

$$=.1.$$

EXERCISES

Group A

Rationalize the denominator and simplify:

1. $\dfrac{6}{\sqrt{3}+\sqrt{2}}$.
2. $\dfrac{1}{\sqrt{5}-\sqrt{2}}$.
3. $\dfrac{2\sqrt{2}}{\sqrt{8}-\sqrt{2}}$.
4. $\dfrac{6-\sqrt{3}}{2-\sqrt{3}}$.
5. $\dfrac{1+\sqrt{2}}{1-\sqrt{2}}$.
6. $\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$.
7. $\dfrac{2\sqrt{5}-\sqrt{3}}{3\sqrt{5}-5}$.
8. $\dfrac{2\sqrt{3}+3\sqrt{2}}{\sqrt{6}+\sqrt{3}}$.
9. $\dfrac{2\sqrt{10}-5}{\sqrt{5}+2\sqrt{2}}$.
10. $\dfrac{\sqrt{6}-\sqrt{3}}{4-2\sqrt{2}}$.
11. $\dfrac{2\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.
12. $\dfrac{4}{\sqrt{a}-\sqrt{b}}$.
13. $\dfrac{a\sqrt{b}}{\sqrt{b}+c}$.
14. $\dfrac{x-y}{\sqrt{x}+\sqrt{y}}$.

Group B

Find the value of each of the following correct to the nearest tenth:

15. $\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$.

16. $\dfrac{4\sqrt{7}}{\sqrt{7}-\sqrt{5}}$.

17. $\dfrac{8-\sqrt{3}}{2-\sqrt{3}}$.

18. $\dfrac{8+3\sqrt{7}}{3+\sqrt{7}}$.

19. $\dfrac{\sqrt{2}+\sqrt{10}}{\sqrt{2}-\sqrt{10}}$.

20. $\dfrac{5}{5-\sqrt{7}}$.

21. $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$.

22. $\dfrac{5-2\sqrt{3}}{2+\sqrt{3}}$.

23. $\dfrac{2+\sqrt{5}}{\sqrt{5}-1}$.

24. $\dfrac{\sqrt{3}-6}{3+2\sqrt{3}}$.

Group C

Simplify:

25. $\dfrac{m+\sqrt{m-1}}{m-\sqrt{m-1}}$.

26. $\dfrac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$.

27. Multiply the numerator and the denominator of the fraction $\dfrac{2+\sqrt{3}+\sqrt{5}}{2+\sqrt{3}-\sqrt{5}}$ by $2+\sqrt{3}+\sqrt{5}$. Then rationalize the denominator of the resulting fraction.

28. Find the value of $\dfrac{a-b}{a^2-b^2}$ when $a=\sqrt{5}+1$ and $b=\sqrt{3}-1$.

29. Find the value of $\dfrac{a^2-ab+b^2}{a^3+b^3}$ if $a=\sqrt{2}+1$ and $b=\sqrt{3}-1$.

30. In the formula $s=\dfrac{a}{1-r}$ find the value of s correct to the nearest tenth if $a=1+\sqrt{3}$ and $r=2-\sqrt{3}$.

Radical Equations

Problem. The greatest distance from which a lighthouse can be seen at sea is found by the formula $d = \sqrt{\frac{3h}{2}}$, in which d represents the distance in miles, and h, the height in feet of the lighthouse above sea level. If the greatest distance at sea from which a certain lighthouse can be seen is 30 miles, how many feet above sea level is the lighthouse?

Evidently the equation will be $30 = \sqrt{\frac{3h}{2}}$. An equation such as this, in which the unknown quantity appears under the radical sign, is called a *radical equation.*

We have just learned that multiplying the square root of a quantity by itself removes the radical sign; *i.e.*, if the square root of a quantity is squared, the radical sign vanishes. But if we square the right-hand member of the equation, we must square the left-hand member of the equation. Why?

Thus, squaring both members of the equation $30 = \sqrt{\frac{3h}{2}}$, we have

$$900 = \frac{3h}{2}.$$

$$\therefore h = 600 \text{ ft.}$$

Thus, in general, the method of solving radical equations is: Square both members of the equation if the root index is 2, cube both members if the root index is 3, and so on. It is desirable, however, to have one radical alone on one side of the equation, before squaring or cubing.

How extraneous roots occur. Let us consider the equation $x = a$. This equation, being a linear equation, has one root. Now let us square both sides of the equation and we have $x^2 = a^2$, a quadratic equation, which must have two roots, namely, $+a$ and $-a$. You will observe that the first root will satisfy the equation $x = a$, whereas the second will not. This root $-a$ is an extraneous root and was introduced by squaring both sides of the equation. Thus squaring both members of an equation introduces a new root.

Again let us consider the equation $x - 2 = 0$. Here again we have a linear equation with one root, 2. Now multiply both sides of the equation by x, and we have $x^2 - 2x = 0$, a quadratic equation which has two roots, 2 and 0. The first root checks in the equation $x - 2 = 0$, but the second does not, and is therefore extraneous. Thus you will observe that multiplying both members of an equation by an expression involving the unknown quantity frequently introduces an extraneous root. *Great care must, therefore, be exercised to check all the roots of a radical equation, as extraneous roots may have been introduced.*

Illustrative examples.

Example 1. Solve and check $\sqrt{x+2} + x = 4$.

Solution

Transposing	$\sqrt{x+2} = 4 - x.$
Squaring both sides	$(\sqrt{x+2})^2 = (4-x)^2.$
Simplifying	$x + 2 = 16 - 8x + x^2.$
Collecting like terms	$x^2 - 9x + 14 = 0.$
Factoring	$(x-2)(x-7) = 0.$
Solving	$x - 2 = 0.$ $\quad\vert\quad$ $x - 7 = 0.$
	$x = +2.$ $\quad\vert\quad$ $x = +7.$

Check

$x = 2.$	$x = 7.$
$\sqrt{2+2} + 2 = 4.$	$\sqrt{7+2} + 7 = 4.$
$2 + 2 = 4.$	$3 + 7 = 4.$
$\therefore$ 2 is a root.	$\therefore$ 7 is extraneous.

Note that the root 7 would have checked if the sign of the 3 had been minus instead of plus. In this way an extraneous root can generally be distinguished from the result of an error, for the extraneous root would have checked, if at this point the sign of some number had been plus instead of minus or minus instead of plus. This would rarely happen as the result of an error.

Example 2. Solve and check $5\sqrt{x-6} = x$.

Solution

Squaring both members
$$(5\sqrt{x-6})^2 = (x)^2$$
$$25(x-6) = x^2$$
$$25x - 150 = x^2.$$

Rearranging terms
$$x^2 - 25x + 150 = 0.$$

Factoring
$$(x-15)(x-10) = 0.$$

Solving
$$x - 15 = 0. \qquad x - 10 = 0.$$
$$x = 15. \qquad x = 10.$$

Check

$x = 15.$	$x = 10.$
$5\sqrt{x-6} = x.$	$5\sqrt{x-6} = x.$
$5\sqrt{9} = 15.$	$5\sqrt{4} = 10.$
$15 = 15.$	$10 = 10.$
$\therefore$ 15 is a root.	$\therefore$ 10 is a root.

Example 3. Solve $\sqrt{x+5} + \sqrt{x} = 1$.

Solution

Transposing
$$\sqrt{x+5} = 1 - \sqrt{x}.$$

Squaring both sides
$$(\sqrt{x+5})^2 = (1 - \sqrt{x})^2$$
$$x + 5 = 1 - 2\sqrt{x} + x.$$

Combining like terms
$$2\sqrt{x} = -4.$$

Dividing both sides by 2
$$\sqrt{x} = -2.$$

Squaring both sides
$$x = 4.$$

Check

Substituting $x = 4$.
$$\sqrt{x+5} + \sqrt{x} = 1.$$
$$\sqrt{4+5} + \sqrt{4} = 1.$$
$$3 + 2 = 1.$$

$\therefore$ 4 is extraneous.

Example 4. Solve $\sqrt[3]{x+6} = 2.$

Solution	*Check*
Cubing both sides	$x = 2.$
$(\sqrt[3]{x+6})^3 = (2)^3.$	$\sqrt[3]{x+6} = 2.$
$x + 6 = 8.$	$\sqrt[3]{8} = 2.$
$x = 2.$	$2 = 2.$
	$\therefore$ 2 is a root.

EXERCISES

Solve each of the following equations, check, and indicate clearly the extraneous roots:

Group A

1. $\sqrt{x} = 3.$
2. $\sqrt[3]{x} = 2.$
3. $\sqrt{x} - 2 = 0.$
4. $\sqrt[3]{x} + 2 = 0.$
5. $\sqrt{x} + 3 = 2.$
6. $\sqrt[3]{x} - 1 = 2.$
7. $\sqrt[3]{x+2} = 2.$
8. $\sqrt{x-2} = -3.$
9. $\sqrt{x+2} - 2 = 3.$
10. $\sqrt{x-1} + 4 = 3.$
11. $3\sqrt{x} = 10.$
12. $3\sqrt[3]{x} = 2.$
13. $2\sqrt{x} + 1 = 5.$
14. $2\sqrt[3]{x} - 1 = 0.$
15. $5\sqrt{x-3} - 4 = 6.$
16. $2\sqrt[3]{x+4} + 1 = 5.$
17. $\sqrt{2x} = \sqrt{5}.$
18. $\sqrt[3]{2x} = \sqrt[3]{5}.$
19. $\sqrt{2x+5} = \sqrt{3x}.$
20. $\sqrt[3]{2x+7} = \sqrt[3]{3x}.$
21. $\sqrt{x-1} = \sqrt{2x-11}.$
22. $3\sqrt{x} = 2\sqrt{7}.$
23. $2\sqrt[3]{x} = \sqrt[3]{16}.$
24. $2\sqrt{x+4} = \sqrt{5x+4}.$
25. $2\sqrt[3]{2x+5} = 3\sqrt[3]{x-3}.$
26. $\sqrt{x^2+7} = 4.$
27. $\sqrt{x^2+3x+7} = 5.$
28. $\sqrt{3x^2+5x+1} - 3 = 0.$
29. $\sqrt[3]{x^2+2} = 3.$
30. $\sqrt[3]{x^2+2x+3} = 3.$

Group B

31. $\sqrt{x^2-7} = x - 1.$
32. $\sqrt{12x+1} = 3x - 1.$
33. $x + 3 = \sqrt{2x^2-7}.$
34. $\sqrt{x(x+3)} + x = 5.$
35. $\sqrt{2x+7} = \sqrt{x} + 2.$
36. $\sqrt{x-3} + \sqrt{x+4} = 7.$

Solve the formula:

37. $V = \sqrt{2\,gS}$ for S.

38. $A = \dfrac{3\,P}{\sqrt{h}}$ for h.

39. $c = \sqrt{a^2 + b^2}$ for b.

40. $r = \sqrt{\dfrac{a}{\pi}}$ for a.

41. $t = \pi\sqrt{\dfrac{l}{g}}$ for l.

42. $r = \sqrt[3]{\dfrac{3\,V}{4\,\pi}}$ for V.

Group C

Solve, check, and clearly indicate the extraneous roots:

43. $\sqrt{x+2} + \sqrt{x-3} = \sqrt{4\,x-3}$.

44. $\sqrt{3(x+1)} - \sqrt{x-1} = \sqrt{x+2}$.

45. $\sqrt{2\,x+1} = 2\sqrt{x} - \sqrt{x-3}$.

46. $\sqrt{x+3} + \sqrt{x} = \dfrac{5}{\sqrt{x}}$.

47. $\sqrt{x-7} + \dfrac{6}{\sqrt{x+1}} = \sqrt{x+1}$.

48. $6 - \sqrt{4\,x+1} = \dfrac{5}{\sqrt{4\,x+1}}$.

49. $\sqrt{13 + \sqrt{5 + \sqrt{x+2}}} = 4$.

Solve the formula:

50. $\sin A = \pm\sqrt{1 - \cos^2 A}$ for $\cos A$.

51. $\sin \frac{1}{2} A = \pm\sqrt{\dfrac{1 - \cos A}{2}}$ for $\cos A$.

52. $a = \sqrt{b^2 + c^2 - 2\,bc \cos A}$ for $\cos A$.

53. $\tan \frac{1}{2} A = \pm\sqrt{\dfrac{1 - \cos A}{1 + \cos A}}$ for $\cos A$.

Imaginary Numbers

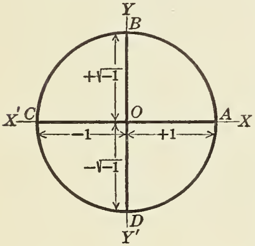
In our previous work we solved the equation $x^2 = -4$ and found that its solution gave rise to a new kind of number, $\sqrt{-4}$, an imaginary number. At that time we defined an imaginary number as an indicated even root of a negative number. We shall now study the imaginary number and gain a deeper insight and understanding of it.

Graphic representation of $\sqrt{-1}$. From the equation $x^2 = -1$, where $x = \sqrt{-1}$, we know that $\sqrt{-1} \cdot \sqrt{-1} = -1$. We know that

$$+1 \cdot \sqrt{-1} = \sqrt{-1}$$

$$+1 \cdot \sqrt{-1} \cdot \sqrt{-1} = -1.$$

That is, if $+1$ is multiplied once by $\sqrt{-1}$, we obtain $\sqrt{-1}$; if $+1$ is multiplied twice in succession by $\sqrt{-1}$, we obtain -1.



These facts may be pictured as in the adjoining diagram. OA represents $+1$ and OC represents -1. OA may be moved into position OC by rotating it through an angle of 180°. But $+1$ may also be changed into -1 by multiplying it twice by $\sqrt{-1}$. Therefore, we can say that a multiplication of $+1$ by $\sqrt{-1}$ twice in succession corresponds to a rotation of $+1$ through an angle of 180°.

From this we can infer that a single multiplication of $+1$ by $\sqrt{-1}$ corresponds to a rotation of $+1$ through an angle of 90°. Therefore, the result of $+1 \cdot \sqrt{-1}$ may be represented by the line OB.

In general, each rotation of $+1$ through 90° is equivalent to a multiplication by $\sqrt{-1}$.

Summarizing, we have:

ROTATION FROM	CORRESPONDS TO THE MULTIPLICATION OF	RESULT
0° to 90°	$+1$ by $\sqrt{-1}$	$=\sqrt{-1}$
90° to 180°	$\sqrt{-1}$ by $\sqrt{-1}$	$=-1$
180° to 270°	-1 by $\sqrt{-1}$	$=-\sqrt{-1}$
270° to 360° (0°)	$-\sqrt{-1}$ by $\sqrt{-1}$	$=+1$

The use of i to stand for $\sqrt{-1}$. The $\sqrt{-1}$ is called the imaginary unit of the system of imaginary numbers and is conveniently represented by i. When working with imaginary numbers, you will find it very convenient to transform them in terms of i. Thus

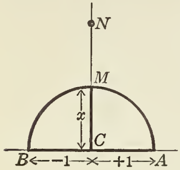
$$\sqrt{-9}=\sqrt{9\times -1}=3\sqrt{-1}=3\,i,$$
$$\sqrt{-25}=\sqrt{25\times -1}=5\sqrt{-1}=5\,i,$$
$$\sqrt{-a^2}=\sqrt{a^2\times -1}=a\sqrt{-1}=ai,$$
$$\sqrt{-3}=\sqrt{3\times -1}=\sqrt{3}\times\sqrt{-1}=i\sqrt{3},$$
$$\sqrt{-18}=\sqrt{9\times 2\times -1}=\sqrt{9}\times\sqrt{2}\times\sqrt{-1}=3\,i\sqrt{2}.$$

EXERCISES

Write each of the following in terms of the imaginary unit i.

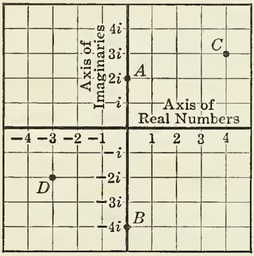
1. $\sqrt{-4}$.
2. $\sqrt{-16}$.
3. $\sqrt{-36}$.
4. $\sqrt{-49}$.
5. $\sqrt{-81}$.
6. $\sqrt{-a^4}$.
7. $\sqrt{-a^6}$.
8. $\sqrt{-a^2b^2}$.
9. $\sqrt{-9\,a^2}$.
10. $\sqrt{-16\,b^4}$.
11. $\sqrt{-2}$.
12. $\sqrt{-5}$.
13. $\sqrt{-7}$.
14. $\sqrt{-8}$.
15. $\sqrt{-50}$.
16. $\sqrt{-27}$.
17. $\sqrt{-72}$.
18. $\sqrt{-8\,x^2}$.
19. $\sqrt{-50\,x^3}$.
20. $\sqrt{-32\,a^5}$.
21. $\sqrt{-63\,y^5}$.

A geometric interpretation of imaginary numbers. Consider the geometric problem: Construct the mean proportional between $+1$ and -1. Evidently our problem is to construct x in the proportion $+1 : x = x : -1$. Constructing the mean proportional as illustrated by the adjoining diagram, we have CM as the required line.



But the proportion $+1 : x = x : -1$ may be written $x^2 = -1$, whence $x = \sqrt{-1}$. Thus $CM(x)$ is equal to $\sqrt{-1}$, that is, to i. In a similar manner CN could be shown to be the mean proportional between $+2$ and -2 and thus equivalent to $2i$.

The vertical axis or in general the y-axis is seen to be an axis upon which imaginary units can be represented, in the same way that real numbers are represented along the horizontal or x-axis. Thus i, $2i$, $3i$, etc. may be represented graphically by laying off unit distances on the *vertical axis* and the real numbers as usual on the horizontal axis.



In the adjoining figure, point A represents $2i$ or $2\sqrt{-1}$; point B represents $-4i$ or $-\sqrt{-16}$. These imaginary numbers are termed *pure imaginary numbers* whereas numbers such as $2 + i$, $3 - 2\sqrt{-1}$, etc., which are partly real and partly imaginary, are called *complex numbers.*

Thus if we wish to represent graphically the complex number $4 + 3i$, we count 4 units to the right on the axis of real numbers, and then 3 units up on the axis of pure imaginaries. The complex number $4 + 3i$ is represented by point C. Similarly, point D represents the complex number $-3 - 2i$.

Addition and subtraction of imaginary numbers.

Illustrative examples.

Example 1. Find the sum of $\sqrt{-9}$, $\sqrt{-25}$, $\sqrt{-36}$.

Solution

$$
\begin{aligned}
&\sqrt{-9}+\sqrt{-25}+\sqrt{-36}\\
&=\sqrt{9\times(-1)}+\sqrt{25\times(-1)}+\sqrt{36\times(-1)}\\
&=3\sqrt{-1}+5\sqrt{-1}+6\sqrt{-1}\\
&=3\,i+5\,i+6\,i=14\,i.
\end{aligned}
$$

Example 2. Add $\sqrt{-20}+\sqrt{-45}-\sqrt{-80}$.

Solution

$$
\begin{aligned}
&\sqrt{-20}+\sqrt{-45}-\sqrt{-80}\\
&=\sqrt{4\times 5\times(-1)}+\sqrt{9\times 5\times(-1)}-\sqrt{16\times 5\times(-1)}\\
&=2\sqrt{5}\sqrt{-1}+3\sqrt{5}\sqrt{-1}-4\sqrt{5}\sqrt{-1}\\
&=2\,i\sqrt{5}+3\,i\sqrt{5}-4\,i\sqrt{5}=i\sqrt{5}.
\end{aligned}
$$

EXERCISES

Simplify:

1. $\sqrt{-4}+\sqrt{-9}$.
2. $\sqrt{-4}-\sqrt{-25}$.
3. $\sqrt{-4}+\sqrt{-49}+\sqrt{-81}$.
4. $\sqrt{-16}-\sqrt{-64}+\sqrt{-100}$.
5. $\sqrt{-x^2}+\sqrt{-25\,x^2}-\sqrt{-4\,x^2}$.
6. $\sqrt{-4\,x^2}-\sqrt{-9\,x^2}+\sqrt{-16\,x^2}$.
7. $\sqrt{-3}+\sqrt{-27}$.
8. $\sqrt{-20}+\sqrt{-45}$.
9. $2\sqrt{-12}+5\sqrt{-27}-\sqrt{-48}$.
10. $4\sqrt{-72}+\sqrt{-32}-2\sqrt{-50}$.
11. $3\sqrt{-12}-4\sqrt{-48}+\sqrt{-75}$.
12. $4\sqrt{-50}+\sqrt{-8}+\sqrt{-72}$.
13. $3\sqrt{-96}+2\sqrt{-54}-\sqrt{-150}$.
14. $\sqrt{-3\,x^2}+2\sqrt{-27\,x^2}+3\sqrt{-12\,x^2}$.
15. $4\sqrt{-5\,x^2}-\sqrt{-45\,x^2}+3\sqrt{-20\,x^2}$.
16. $3\sqrt{-12\,y^3}+4\sqrt{-27\,y^3}-2\sqrt{-48\,y^3}$.

Multiplication and division of imaginary numbers. In our work with radicals you learned that $\sqrt{2} \times \sqrt{3} = \sqrt{6}$, according to the law $\sqrt{a}\sqrt{b} = \sqrt{ab}$. If we apply this principle to the problem $\sqrt{-3} \times \sqrt{-3}$, we seem to obtain $\sqrt{(-3)(-3)} = \sqrt{9} = +3$. This positive value, 3, is evidently incorrect because $\sqrt{+3}$ is not $\sqrt{-3}$. You have also learned that if a square root of a quantity is squared, the radical sign vanishes. That is, $\sqrt{-3} \times \sqrt{-3} = (\sqrt{-3})^2 = -3$. This negative value is evidently correct because the square root of -3 is $\sqrt{-3}$. (However, the use of the imaginary unit i will avoid the introduction of such errors as you have just noted.)

Therefore, in order to operate with imaginary numbers, it is advisable to first write each imaginary number in terms of i. Before applying this process to $\sqrt{-3} \cdot \sqrt{-3}$, let us introduce you to a very helpful table of powers of i.

i		$= \sqrt{-1}$
i^2	$= i \cdot i = \sqrt{-1} \cdot \sqrt{-1}$	$= -1$
i^3	$= i^2 \cdot i = -1 \cdot \sqrt{-1} = -\sqrt{-1}$	$= -i$
i^4	$= i^2 \cdot i^2 = (-1) \cdot (-1)$	$= +1$
i^5	$= i^4 \cdot i = (+1) \cdot \sqrt{-1} = \sqrt{-1}$	$= i$
i^6	$= i^4 \cdot i^2 = (+1) \cdot (-1)$	$= -1$
i^7	$= i^6 \cdot i = (-1) \cdot \sqrt{-1} = -\sqrt{-1}$	$= -i$
i^8	$= i^4 \cdot i^4 = (+1) \cdot (+1)$	$= +1$

You will observe that the powers of i occur in an indefinite number of cycles of the four quantities, i, -1, $-i$, and $+1$, and from this fact we can determine any integral power of i.

Thus $\quad i^{34} = i^{32} \cdot i^2 = (i^4)^8 \cdot i^2 = (1)^8 \cdot (-1) = -1$

and $\quad i^{27} = i^{24} \cdot i^3 = (i^4)^6 \cdot i^3 = (1)^6 \cdot (-i) = -i.$

Returning to the example $\sqrt{-3} \cdot \sqrt{-3}$, and transforming into the i form, we have $i\sqrt{3} \cdot i\sqrt{3} = i^2(3) = -3$.

Illustrative examples.

Example 1. Multiply $2\sqrt{-4}$ by $3\sqrt{-9}$.

Solution

$$\begin{aligned} 2\sqrt{-4} \cdot 3\sqrt{-9} &= 2\sqrt{4(-1)} \cdot 3\sqrt{9(-1)} \\ &= 4\,i \cdot 9\,i = 36\,i^2 = -36. \end{aligned}$$

Example 2. Multiply $2 + 3\sqrt{-3}$ by $4 - 5\sqrt{-3}$.

Solution

$$\begin{aligned} 2 + 3\sqrt{-3} &= 2 + 3\,i\sqrt{3} \\ 4 - 5\sqrt{-3} &= \underline{4 - 5\,i\sqrt{3}} \\ & \quad 8 + 12\,i\sqrt{3} \\ & \quad \underline{\quad - 10\,i\sqrt{3} - (15\,i^2 \times 3)} \\ & \quad 8 + 2\,i\sqrt{3} - (-45) \\ &= 8 + 2\,i\sqrt{3} + 45 = 53 + 2\,i\sqrt{3}. \end{aligned}$$

Example 3. Divide 12 by $3\sqrt{-2}$.

Solution

$$\begin{aligned} \frac{12}{3\sqrt{-2}} &= \frac{12}{3\,i\sqrt{2}} = \frac{4}{i\sqrt{2}} \times \frac{i\sqrt{2}}{i\sqrt{2}} = \frac{4\,i\sqrt{2}}{2\,i^2} \\ &= \frac{4\,i\sqrt{2}}{-2} = -2\,i\sqrt{2}. \end{aligned}$$

Example 4. Simplify $\dfrac{2\sqrt{-3}}{1 - \sqrt{-2}}$ by rationalizing the denominator.

Solution

$$\begin{aligned} \frac{2\,i\sqrt{3}}{1 - i\sqrt{2}} &= \frac{2\,i\sqrt{3}}{1 - i\sqrt{2}} \times \frac{1 + i\sqrt{2}}{1 + i\sqrt{2}} = \frac{2\,i\sqrt{3} + 2\,i^2\sqrt{6}}{1 - (2\,i^2)} \\ &= \frac{2\,i\sqrt{3} - 2\sqrt{6}}{1 - (-2)} = \frac{2(i\sqrt{3} - \sqrt{6})}{3}. \end{aligned}$$

EXERCISES

Group A

Perform the indicated operations, and simplify:

1. $2i \cdot 3i$.

2. $5i \cdot 2i^2$.

3. $-5i \cdot i^3$.

4. $i\sqrt{2} \cdot i\sqrt{3}$.

5. $i\sqrt{4} \cdot i\sqrt{3}$.

6. $\sqrt{-3} \cdot \sqrt{-4}$.

7. $\sqrt{-25} \cdot \sqrt{-4}$.

8. $3\sqrt{-5} \cdot 5\sqrt{-2}$.

9. $\sqrt{12} \cdot \sqrt{-3}$.

10. $4\sqrt{-2} \cdot 3\sqrt{-2}$.

11. $(\sqrt{-4})^2$.

12. $(\sqrt{-3})^2$.

13. $(\sqrt{-2})^3$.

14. $(\sqrt{-4})^4$.

15. $(5+i)(3-i)$.

16. $(2+\sqrt{-1})(3+2\sqrt{-1})$.

17. $(3+\sqrt{-2})(2-\sqrt{-2})$.

18. $(6-\sqrt{-5})(2+\sqrt{-3})$.

19. $(6\sqrt{-5}-2)(2-3i\sqrt{2})$.

20. $(a+bi)(a-bi)$.

21. $(3-i)^2$.

22. $(4+3i)^2$.

23. $(3-i\sqrt{2})^2$.

24. $(a+bi)^2$.

Group B

Rationalize the denominators and simplify:

25. $\frac{3}{i}$.

26. $\frac{\sqrt{-6}}{\sqrt{-2}}$.

27. $\frac{12}{\sqrt{-3}}$.

28. $\frac{6i^2}{\sqrt{-3}}$.

29. $\frac{2}{1+i}$.

30. $\frac{8}{1-\sqrt{-1}}$.

31. $\frac{4}{1-2i}$.

32. $\frac{4}{2-\sqrt{-4}}$.

33. $\frac{1-i}{1+i}$.

34. $\frac{4-i}{2-i}$.

35. $\frac{2+3i}{1-2i}$.

36. $\frac{3-\sqrt{-1}}{2+\sqrt{-3}}$.

37. Simplify: $(1+i)^2-(1-i)^2$.

38. Subtract: $\frac{1-i}{1+i}-\frac{1+i}{1-i}$.

39. Add: $\frac{2}{4-i^2}+\frac{3}{2+i}$.

Group C

40. If $x = 2 + \sqrt{-3}$, find the value of $x^2 - 4x + 7$.

41. Show that $2 - \sqrt{-3}$ is a root of $x^2 - 4x + 7 = 0$.

42. Show that $2 + i$ is a root of $x^2 - 4x + 5 = 0$.

43. Show that $\dfrac{-1 + \sqrt{-3}}{2}$ is a root of $x^3 - 1 = 0$.

44. Form the equation whose roots are:

(*a*) $1 + i, 1 - i$.

(*b*) $3 + 2\sqrt{-1}, -3 + 2\sqrt{-1}$.

(*c*) $2 + \sqrt{-3}, 2 - \sqrt{-3}$.

(*d*) $\dfrac{1 + \sqrt{-3}}{2}, \dfrac{1 - \sqrt{-3}}{2}$.

45. Multiply $\cos x + i \sin x$ by $\cos x - i \sin x$.

46. Expand $(\cos x + i \sin x)^2$.

47. If $e^{ix} = \cos x + i \sin x$, find the value of $e^{i\pi}$ by substituting π for x.

To Express the Values of All the Functions in Terms of One Function

Illustrative example. Express, in terms of $\sin A$, the remaining five functions of A.

Solution 1.

Since $\sin^2 A + \cos^2 A = 1$, $\cos^2 A = 1 - \sin^2 A$,

$$\therefore \cos A = \pm\sqrt{1 - \sin^2 A}.$$

Since $\tan A = \dfrac{\sin A}{\cos A}$, $\tan A = \dfrac{\sin A}{\pm\sqrt{1 - \sin^2 A}}$,

$$\therefore \tan A = \pm \frac{\sin A\sqrt{1 - \sin^2 A}}{1 - \sin^2 A}.$$

Since $\cot A = \dfrac{\cos A}{\sin A}$,

$$\therefore \cot A = \pm \frac{\sqrt{1 - \sin^2 A}}{\sin A}.$$

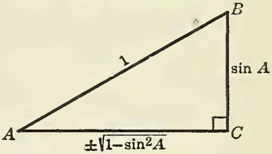
Since $\quad \sec A = \dfrac{1}{\cos A}, \ \sec A = \dfrac{1}{\pm\sqrt{1 - \sin^2 A}},$

$$\therefore \sec A = \pm\frac{\sqrt{1 - \sin^2 A}}{1 - \sin^2 A}.$$

Since $\quad \csc A = \dfrac{1}{\sin A},$

$$\therefore \csc A = \frac{1}{\sin A}.$$

Solution 2. In the right triangle ABC, let the denominator (AB) of the sine ratio be unity. Then the numerator, BC, is $\sin A$. By the Pythagorean theorem, we find the third side, AC, to be $\pm\sqrt{1 - \sin^2 A}$. Now from the figure, we can read off:



$$\cos A = \pm\sqrt{1 - \sin^2 A}.$$

$$\tan A = \pm\frac{\sin A}{\sqrt{1 - \sin^2 A}}. \qquad \sec A = \pm\frac{1}{\sqrt{1 - \sin^2 A}}.$$

$$\cot A = \pm\frac{\sqrt{1 - \sin^2 A}}{\sin A}. \qquad \csc A = \frac{1}{\sin A}.$$

EXERCISES

Group A

1. Express the other five functions of x in terms of:
(a) $\cos x$; (b) $\tan x$; (c) $\cot x$; (d) $\sec x$; (e) $\csc x$.

2. Given: $\sin x = a$, x being in Q II. Find the values of the other five functions of x in terms of a.

3. Given: $\tan x = \dfrac{b}{a}$, x being in Q III. Find the values of the other five functions of x in terms of b and a.

Prove the following identities:

4. $\tan A\sqrt{1 - \sin^2 A} = \sin A$.
5. $\cot A\sqrt{1 - \cos^2 A} = \cos A$.
6. $\sin A\sqrt{1 + \tan^2 A} = \tan A$.
7. $\tan B \cos B = \sqrt{1 - \cos^2 B}$.
8. $\cos^2 \theta - \sin^2 \theta = \sqrt{1 - 4 \sin^2 \theta \cos^2 \theta}$.
9. $\sin A + \cos A = \sqrt{1 + 2 \sin A \cos A}$.

Solve the following equations for all values of the unknown between 0° and 360°. Check your solutions.

10. $\sqrt{\dfrac{\sin \theta}{2}} = \sin \theta$.
11. $2 \cos x = \sqrt{8 \sin x - 1}$.
12. $2 \sin x = \sqrt{1 + 4 \cos x}$.
13. $\cos A = \sqrt{\dfrac{4 + \sin A}{6}}$.
14. $\sec x = \sqrt{7 - 4 \sin^2 x}$.
15. $\sqrt{2 - \cot^2 x} = \tan x$.

Group B

Prove the following identities:

16. $\sec A - \tan A = \dfrac{\sqrt{1 - \sin A}}{\sqrt{1 + \sin A}}$.
17. $\sqrt{\dfrac{1 + \sin y}{1 - \sin y}} = \dfrac{1 + \sin y}{\cos y}$.
18. $\sin x \cos x = \sqrt{\dfrac{1 - \sin^4 x - \cos^4 x}{2}}$.
19. $\sqrt{\dfrac{1 + \sin z}{1 - \sin z}} = \dfrac{\cos z}{1 - \sin z}$.

Solve the following equations for all values of the unknown between 0° and 360°. Check your solutions.

20. $\sqrt{6 - \tan^2 x} - 2 \sin x = 0$.
21. $\cot x + \csc x = \sqrt{3}$.
22. $\sqrt{3 - 2 \cos^2 x} - 2 \sin x = 0$.
23. $\sqrt{1 - \sin A} = \sqrt{2} \cos A$.

Group C

Prove the following identities:

24. $\dfrac{\tan x + 1}{\tan x - 1} = \sqrt{\dfrac{1 + 2 \sin x \cos x}{1 - 2 \sin x \cos x}}.$

25. $\csc y = \sqrt{\dfrac{\csc^2 y - 2}{\cos^2 y - \sin^2 y}}.$

26. $\dfrac{\sqrt{\cos^4 B - \sin^4 B + 1}}{\sqrt{2}} = \cos B.$

27. $\sin \theta \cos \theta = \dfrac{\sqrt{1 - \sin^6 \theta - \cos^6 \theta}}{\sqrt{3}}.$

Solve the following equations for all values of the unknown between 0° and 360°. Check your solutions.

28. $3 \sec^2 y - 2\sqrt{3} \tan y = 6.$

29. $\sin \theta + \sqrt{3} \cos \theta = 2.$

30. $\tan^2 \theta + \sqrt{3} = (1 + \sqrt{3}) \tan \theta.$

31. $\sqrt{1 + \sin x} - \sqrt{1 - \sin x} = 2 \cos x.$

The Radical in Mathematics

The radical sign $\sqrt{\ }$ was introduced by Rudolff (about 1525) in the first textbook in Algebra in the German language. This symbol seems to have been derived from the first letter of the Latin word *radix,* meaning root.

Just as we have seen that negative roots were not understood for hundreds of years, so the square roots of negative numbers were not well understood until it was possible to represent them graphically. This was accomplished by Wessel (1797), who used what we now call the y-axis to represent imaginary numbers. As soon as the meaning of imaginary numbers was thoroughly understood symbolism for them was soon standardized. Euler (1748) was the first to use i for $\sqrt{-1}$.

CUMULATIVE REVIEW

Chapters VIII, IX, and X

1. Which of these statements are true? Which are false?

(*a*) If $x = y^2 + y - 6$, x is a function of y.

(*b*) The graphs of the equations $x = 3y - 6$ and $x = \frac{1}{3}y - 6$ pass through the same point on the x-axis.

(*c*) The equations $5x + y = 6$ and $10x + 2y = 12$ have many common solutions.

(*d*) If in 1 hour a boy can row 5 miles with a current, but only 3 miles when rowing against the same current, the rate of the current is 2 miles an hour.

(*e*) The graph of every quadratic equation having two variables is a parabola.

(*f*) $\sqrt{a(a + b)} = a + b\sqrt{a}$.

(*g*) $\sqrt{36 - 9} = 5$.

(*h*) If $2\sqrt{x} = 3$, then $x = 2.25$.

(*i*) $(a\sqrt{b} - b\sqrt{a})^2 = (b\sqrt{a} - a\sqrt{b})^2$.

(*j*) $\dfrac{8\sqrt{18}}{\sqrt{2}} = 24$.

2. Complete each of the following statements:

(*a*) If a represents the sum and also the product of the roots of a quadratic equation in x, the equation may be written ? .

(*b*) The minimum value of y in the equation $y = x^2 - 4x + 3$ can be found by substituting in it $x =$? .

(*c*) If $\dfrac{2x + y}{2x - y} = \dfrac{7}{2}$, then the value of y in terms of x is ? .

(*d*) The quadratic equation in terms of x only which can be formed from the equations $x^2 + y^2 = 4$ and $y = 2x$ is ? .

(*e*) The smallest positive value of x which will satisfy the equations: $\sin x = y$ and $y = \frac{1}{2}\sqrt{3}$ is ? .

(*f*) The value of $\dfrac{2}{\sqrt{2}}$ correct to the nearest tenth is ? .

(g) To rationalize a fraction whose denominator is $\sqrt{a} - b$, multiply both numerator and denominator by ? .

(h) If $\sqrt[3]{x} = -2$, then $x =$? .

(i) The extraneous root of the equation: $x - \sqrt{x+2} = 0$ is ? .

(j) If $\sqrt[3]{x} = 4$, then $x =$? and $x\sqrt{x} =$? .

3. If a baseball is thrown vertically upward with a velocity of 80 feet per second, the height reached after t seconds is given by the formula $h = 80\,t - 16\,t^2$. Find the greatest height which the ball reaches. After how many seconds will it attain this height?

4. Find the roots of $2\,x^2 - 3\,x = 3$ correct to the nearest tenth.

5. Solve for x and y and check:

$$\begin{aligned} ax - by &= a^2 + b^2 \\ x - y &= 2\,b. \end{aligned}$$

6. The numerator and denominator of a fraction are in the ratio 7 : 8. If, however, 10 be subtracted from each, their ratio will be 6 : 7. Find the original fraction.

7. The sum of the areas of two squares is 100 square inches and the perimeter of the first square exceeds the perimeter of the second square by 8 inches. Find the length of a side of each square.

8. Simplify $3\sqrt{8\,a^3} - a\sqrt{\frac{a}{2}} + 6\,a\sqrt{2\,a} - a\sqrt{\frac{9}{2}\,a}$.

9. Prove that $4 + 3\sqrt{2}$ is a root of the equation $y^2 - 8\,y = 2$.

10. Form the equation whose roots are $3 - \sqrt{5}$ and $3 + \sqrt{5}$.

11. Solve and check $\sqrt{3\,x - 2} = 2 - \sqrt{3\,x}$.

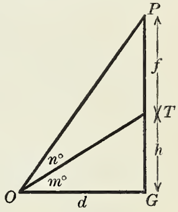
12. Express $\sec x$ and $\cos x$ in terms of $\tan^2 x$.

CHAPTER XI. TRIGONOMETRIC FORMULAS

The great body of scientific knowledge, a great deal of the essential facts of financial and economic sciences, the endless social and political problems are accessible only to those who can think in terms of functional methods and of functional relationships. — J. S. GEORGES.

FUNCTIONS OF THE SUM OF TWO ANGLES

Problem. A flagstaff f feet high stands on the top of a house. From a point of observation in the plane on which the house stands, the house subtends an angle of $m°$ while the flagstaff subtends an angle of $n°$. Find a formula for the height (h) of the house above the point of observation in terms of f, m, and n. Find a formula also for the distance (d) of the observer from the house in terms of f, m, and n.



The answers to this problem can be obtained by solving the following pair of simultaneous equations:

$$\tan m = \frac{h}{d} \tag{1}$$

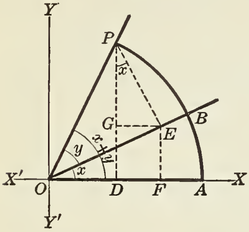
$$\tan (m + n) = \frac{h + f}{d}. \tag{2}$$

Equation (1) contains nothing new, but equation (2) does, namely, the tangent of the sum of two angles. As m and n are angles, $m + n$ is also an angle. But $\tan (m + n)$ is not equal to $\tan m + \tan n$. For example, if $m = 45°$ and $n = 30°$, then $\tan (m + n) = \tan (45° + 30°) = \tan 75° = 3.7321$.
But $\tan m + \tan n = \tan 45° + \tan 30° = 1.0000 + 0.5774$
$= 1.5774$.

In general, then, $\tan (m + n)$ does not equal $\tan m + \tan n$; similarly, $\sin (x + y)$ does not equal $\sin x + \sin y$ nor does $\cos (x + y)$ equal $\cos x + \cos y$. It is clear that we must develop new formulas for the functions of the sum of two angles. We shall begin by deriving new formulas for $\sin (x + y)$ and $\cos (x + y)$ in terms of the functions of x and y.

To derive the formulas for $\sin (x + y)$ and $\cos (x + y)$.

Case 1. Angles x, y, and $(x + y)$ are acute; the angle $(x + y)$ is therefore in Q I.



In $\odot O$, let radius $OA = 1$, $\angle AOB = x$, $\angle BOP = y$, $\angle AOP = (x + y)$.

From P two perpendiculars are drawn,

$$PD \perp OA, \quad PE \perp OB.$$

From E two perpendiculars are drawn,

$$EF \perp OA, \quad EG \perp PD.$$

$\angle GPE = \angle AOB = \angle x$ [acute $\measuredangle$ whose sides are $\perp$ are $=$.]

$$\sin (x + y) = \frac{PD}{OP \text{ or } 1} = PD = EF + PG. \quad (1)$$

Value of EF:

In rt. $\triangle OEF$, $\dfrac{EF}{OE} = \sin x$; *i.e.*, $EF = \sin x \cdot OE$.

But in rt. $\triangle OEP$, $\dfrac{OE}{OP \text{ or } 1} = \cos y$; $\therefore$ $EF = \sin x \cos y$. (2)

Value of PG:

In rt. $\triangle PGE$, $\dfrac{PG}{PE} = \cos x$; *i.e.*, $PG = \cos x \cdot PE$.

But in rt. $\triangle OEP$, $\dfrac{PE}{OP \text{ or } 1} = \sin y$; $\therefore$ $PG = \cos x \sin y$. (3)

Substituting (2) and (3) in (1), we obtain

$$\mathbf{\sin (x + y) = \sin x \cos y + \cos x \sin y.}$$

In a similar manner, using the same diagram,

$$\cos(x + y) = \frac{OD}{OP \text{ or } 1} = OD = OF - DF = OF - GE. \quad (4)$$

Value of OF:

In rt. $\triangle OEF$, $\dfrac{OF}{OE} = \cos x$; *i.e.*, $OF = \cos x \cdot OE$.

But in rt. $\triangle OEP$, $\dfrac{OE}{OP \text{ or } 1} = \cos y$; $\therefore\ OF = \cos x \cos y. \quad (5)$

Value of GE:

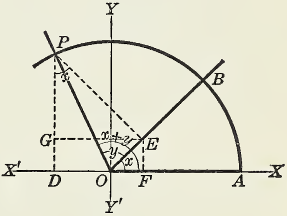
In rt. $\triangle PGE$, $\dfrac{GE}{PE} = \sin x$; *i.e.*, $GE = \sin x \cdot PE$.

But in rt. $\triangle OEP$, $\dfrac{PE}{OP \text{ or } 1} = \sin y$; $\therefore\ GE = \sin x \sin y. \quad (6)$

Substituting (5) and (6) in (4), we obtain

$$\mathbf{\cos(x + y) = \cos x \cos y - \sin x \sin y.}$$

Case 2. Angles x and y are acute, while the angle $(x + y)$ is obtuse and is therefore in Q II.



The construction of the figure and proof of the formula $\sin(x + y) = \sin x \cos y + \cos x \sin y$ are exactly the same, word for word, as in Case 1. Trace this proof, using the diagram at the right.

The proof of the formula $\cos(x + y) = \cos x \cos y - \sin x \sin y$ is the same as that given for Case 1 except that the cosine is negative. Thus, using the adjoining figure,

$$\cos(x + y) = -\frac{OD}{OP} = -\frac{OD}{1} = -OD = -(DF - OF)$$
$$= -(GE - OF) = OF - GE.$$

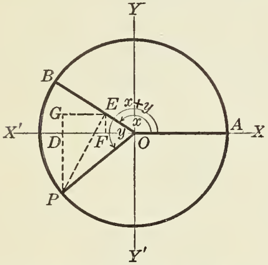
The rest of the proof is the same as in Case 1.

The formulas for sin $(x + y)$ and cos $(x + y)$ have now been shown to be true for all positive acute angles x and y, since the sum of two positive acute angles is an angle which must lie in either quadrant I or II.

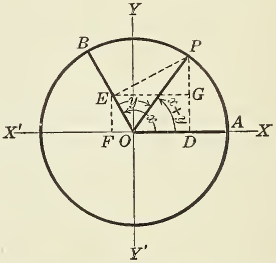
The discussion and proofs for all other *cases* in which the angle $(x + y)$ terminates in any quadrant are left to the student. The diagrams for two of these cases are given below.

If time does not permit, this work may be omitted, and the general truth of the formulas for sin $(x + y)$ and cos $(x + y)$ may be assumed without further proof.

Case 3. x is obtuse, y is acute
angle $(x + y)$ is in Q III.



Case 4. x is obtuse, y is acute and negative
angle $(x + y)$ is in Q I.



Thus the formulas for sin $(x + y)$ and cos $(x + y)$ hold true in all quadrants, for all values of x and y, positive and negative. They are often referred to as the *addition formulas* for the sine and cosine.

Formula for tan $(x + y)$. By our previous work, we have, for all values of x and y,

$$\tan (x + y) = \frac{\sin (x + y)}{\cos (x + y)}$$

$$= \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}. \qquad (1)$$

The right-hand member of equation (1) can be expressed in terms of tangents if we divide both numerator and denominator by $\cos x \cos y$. Thus we obtain,

$$\tan(x+y) = \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}{1 - \dfrac{\sin x}{\cos x} \cdot \dfrac{\sin y}{\cos y}}.$$

$$\therefore \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

Formula for cot $(x + y)$. Again, from our previous work, we have, for all values of x and y,

$$\cot(x+y) = \frac{\cos(x+y)}{\sin(x+y)}$$

$$= \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y}. \qquad (1)$$

The right-hand member of equation (1) can be expressed in terms of cotangents if we divide both numerator and denominator by $\sin x \sin y$. Thus we obtain,

$$\cot(x+y) = \frac{\dfrac{\cos x \cos y}{\sin x \sin y} - \dfrac{\sin x \sin y}{\sin x \sin y}}{\dfrac{\sin x \cos y}{\sin x \sin y} + \dfrac{\cos x \sin y}{\sin x \sin y}}$$

$$= \frac{\dfrac{\cos x}{\sin x}\dfrac{\cos y}{\sin y} - 1}{\dfrac{\cos y}{\sin y} + \dfrac{\cos x}{\sin x}}.$$

$$\therefore \cot(x+y) = \frac{\cot x \cdot \cot y - 1}{\cot y + \cot x}.$$

Remember

1. $\sin(x+y) = \sin x \cos y + \cos x \sin y.$
2. $\cos(x+y) = \cos x \cos y - \sin x \sin y.$
3. $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}.$
4. $\cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot y + \cot x}.$

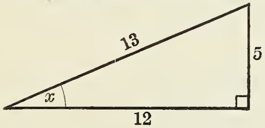
Illustrative examples.

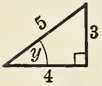
Example 1. Prove by the formula for $\sin(x+y)$ that $\sin(90° + x) = \cos x$.

Solution

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$
$$\text{Let } x = 90°,\ y = x.$$
$$\sin(90° + x) = \sin 90° \cos x + \cos 90° \sin x$$
$$= (1)(\cos x) + (0)\sin x.$$
$$\therefore \sin(90° + x) = \cos x.$$

Example 2. If x and y are acute angles and $\sin x = \frac{5}{13}$ and $\sin y = \frac{3}{5}$, find the value of $\cos(x+y)$.





Solution

$$\sin x = \frac{5}{13} = \frac{\text{ordinate}}{\text{distance}}.$$

$\therefore$ third side of basic triangle

$$= \sqrt{13^2 - 5^2} = 12.$$

$$\sin y = \frac{3}{5} = \frac{\text{ordinate}}{\text{distance}}.$$

$\therefore$ third side of basic triangle $= \sqrt{5^2 - 3^2} = 4.$

$$\text{But } \cos(x+y) = \cos x \cos y - \sin x \sin y.$$
$$= \tfrac{12}{13} \cdot \tfrac{4}{5} - \tfrac{5}{13} \cdot \tfrac{3}{5}$$
$$= \tfrac{48}{65} - \tfrac{15}{65} = \tfrac{33}{65}.$$

Example 3. Find the value of sin 75° in radical form.

Solution

$$\begin{aligned} \sin 75^\circ &= \sin(45^\circ + 30^\circ) \\ &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \\ &= (\tfrac{1}{2}\sqrt{2})(\tfrac{1}{2}\sqrt{3}) + (\tfrac{1}{2}\sqrt{2})(\tfrac{1}{2}) \\ &= \tfrac{1}{4}\sqrt{6} + \tfrac{1}{4}\sqrt{2} \\ &= \tfrac{1}{4}(\sqrt{6} + \sqrt{2}). \end{aligned}$$

Example 4. Solve for all values of x between 0° and 360°:

$$\sin\left(x + \frac{\pi}{6}\right) - \cos\left(x + \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}.$$

Solution

$$\sin\left(x + \frac{\pi}{6}\right) - \cos\left(x + \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}.$$

$$\sin(x + 30^\circ) - \cos(x + 60^\circ) = -\frac{\sqrt{3}}{2}.$$

$$(\sin x \cos 30^\circ + \cos x \sin 30^\circ) - (\cos x \cos 60^\circ - \sin x \sin 60^\circ) = -\frac{\sqrt{3}}{2}.$$

$$\sin x \cdot \tfrac{1}{2}\sqrt{3} + \cos x \cdot \tfrac{1}{2} - \cos x \cdot \tfrac{1}{2} + \sin x \cdot \tfrac{1}{2}\sqrt{3} = -\frac{\sqrt{3}}{2}.$$

$$(\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{3}) \cdot \sin x = -\frac{\sqrt{3}}{2}.$$

$$\sqrt{3}\sin x = -\frac{\sqrt{3}}{2}.$$

$$\sin x = -\tfrac{1}{2}.$$

[The angle whose sine is $\frac{1}{2}$ is 30°. But since $\sin x$ is negative, one value of x is in Q III, $(180^\circ + x)$, and another is in Q IV, $(360^\circ - x)$.]

$$x_1 = 180^\circ + 30^\circ = 210^\circ$$

and $$x_2 = 360^\circ - 30^\circ = 330^\circ.$$

$$\therefore x = 210^\circ, 330^\circ.$$

Check

Substituting $x = 210°$.

$$\sin(210° + 30°) - \cos(210° + 60°) = -\frac{\sqrt{3}}{2}.$$

$$\sin 240° - \cos 270° = -\frac{\sqrt{3}}{2}.$$

$$-\sin 60° - 0 = -\frac{\sqrt{3}}{2}.$$

$$-\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}.$$

Substituting $x = 330°$.

$$\sin(330° + 30°) - \cos(330° + 60°) = -\frac{\sqrt{3}}{2}.$$

$$\sin 360° - \cos 390° = -\frac{\sqrt{3}}{2}.$$

$$0 - \cos 30° = -\frac{\sqrt{3}}{2}.$$

$$-\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}.$$

To the suggestions made on page 77 for proving an identity, the following may now be added: *If an identity contains functions of the sum of two angles, expand such functions by means of the addition formulas into functions of single angles.*

EXERCISES

1. Expand each of the following by means of the appropriate addition formula.

(*a*) $\cos(x + 45°)$. (*c*) $\cot(30° + y)$. (*e*) $\sin\left(\frac{\pi}{2} + y\right)$.

(*b*) $\tan(x + 45°)$. (*d*) $\sin(60° + y)$. (*f*) $\tan\left(A + \frac{\pi}{4}\right)$.

2. Show that: (*a*) $\sin(x + 45°) = \frac{1}{2}\sqrt{2}(\sin x + \cos x)$.

(*b*) $\cos(60° + y) = \frac{1}{2}(\cos y - \sqrt{3}\sin y)$.

3. Show that: $\tan(A + 60°) = \dfrac{\tan A + \sqrt{3}}{1 - \sqrt{3}\tan A}$.

4. If $\sin x = \frac{12}{13}$ and x is acute, find the value of each of the following:

(a) $\sin (90° + x)$. (c) $\cot (30° + x)$. (e) $\cos (x + x)$.
(b) $\cos (60° + x)$. (d) $\sin (x + x)$. (f) $\tan (x + x)$.

5. If $\cot A = 5$ and $\cot B = 4$, find $\cot (A + B)$.

6. If $\sin x = \frac{4}{5}$, $\cos y = \frac{12}{13}$, x and y being acute, find the value of:

(a) $\sin (x + y)$. (c) $\tan (x + y)$.
(b) $\cos (x + y)$. (d) $\cot (x + y)$.

7. (a) If $\tan B = 1$ and $\tan C = 0.5$, find the value of $\tan (B + C)$.

(b) If B and C are in Q I, in which quadrant will the angle $(B + C)$ lie?

(c) Find the number of degrees in $(B + C)$.

8. If $\sin A = \frac{3}{5}$, and angle A is in Q II, find the value of $\sin\left(A + \frac{\pi}{4}\right)$.

9. If $\tan A = \frac{5}{6}$ and $\tan B = \frac{1}{11}$, find the value of $\tan (A + B)$. How many degrees are there in $(A + B)$?

10. If $\sin x = \frac{4}{5}$, x being in Q II, and $\tan y = \frac{5}{12}$, y being in Q I, find the value of:

(a) $\sin (x + y)$. (c) $\tan (x + y)$.
(b) $\cos (x + y)$. (d) $\cot (x + y)$.

11. If $\sin A = -\frac{3}{5}$, A being in Q III, and $\cos B = \frac{5}{13}$, B being in Q IV, find the value of $\cos (A + B)$.

Prove the following identities:

12. $\sin (x + 60°) - \cos (x + 30°) = \sin x$.

13. $\sin (x + y) \cos y - \cos (x + y) \sin y = \sin x$.

14. $\tan (45° + x) = \dfrac{\cos x + \sin x}{\cos x - \sin x}$.

15. $\tan A + \tan B = \dfrac{\sin (A + B)}{\cos A \cos B}$.

16. $\sin(\theta + 30°) = \dfrac{\cos\theta + \sqrt{3}\sin\theta}{2}.$

17. $\tan(\theta + 45°) = \dfrac{1 + \tan\theta}{1 - \tan\theta}.$

18. $\dfrac{\sin(A + B)\cos C}{\sin(A + C)\cos B} = \dfrac{1 + \cot A \tan B}{1 + \cot A \tan C}.$

Solve the following equations for all values of x between 0° and 360°. Check.

19. $\cos(x + 30°) = \sin x.$ **20.** $\sin(x + 30°) = \cos x.$

21. $2\sin(x + 120°) + 2\sin(x + 60°) = 3.$

22. $\sin x = 2\sin\left(x + \dfrac{\pi}{3}\right).$

Functions of the Difference of Two Angles

To derive the formulas for sin $(x - y)$ and cos $(x - y)$. We have already shown that the formulas

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \tag{1}$$

and

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \tag{2}$$

hold true for values of x and y of any size, positive or negative. Therefore, since (1) holds whether y is positive or negative, we may substitute $-y$ for y, so that we obtain

$$\sin[x + (-y)] = \sin x \cos(-y) + \cos x \sin(-y). \tag{3}$$

But $\cos(-y) = \cos y$ and $\sin(-y) = -\sin y.$ (4)

∴ substituting (4) in (3), we have

$$\sin(x - y) = \sin x \cos y - \cos x \sin y.$$

Similarly, substituting $-y$ for y in (2), we obtain

$$\cos[x + (-y)] = \cos x \cos(-y) - \sin x \sin(-y). \tag{5}$$

But $\cos(-y) = \cos y$ and $\sin(-y) = -\sin y.$ (6)

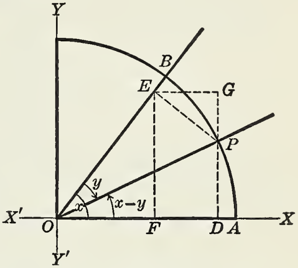
∴ Substituting (6) in (5), we have

$$\cos(x - y) = \cos x \cos y + \sin x \sin y.$$

Observe that the formulas for $\sin(x-y)$ and $\cos(x-y)$ are merely other forms of the formulas for $\sin(x+y)$ and $\cos(x+y)$. Therefore, since the latter formulas are true universally, the former are also true universally, that is, true for all values of x and y, positive or negative.

The formulas for $\sin(x-y)$ and $\cos(x-y)$ are often referred to as the *subtraction* formulas for the sine and cosine.

Geometric proof of formulas for $\sin(x-y)$ and $\cos(x-y)$. x, y, and $(x-y)$ are acute angles.



In $\odot O$, let radius $OA = 1$,

$\angle AOB = x$, $\angle BOP = y$,

$\angle AOP = (x-y)$.

From P two perpendiculars are drawn,

$PD \perp OA$, $PE \perp OB$.

From E two perpendiculars are drawn,

$EF \perp OA$, $EG \perp DP$ produced.

$\angle GPE = \angle BOA = \angle x$. [Acute ∠s whose sides are $\perp$ are $=$.]

$$\sin(x-y) = \frac{PD}{OP \text{ or } 1} = PD = DG - PG = EF - PG. \quad (1)$$

Value of EF:

In rt. $\triangle OEF$, $\dfrac{EF}{OE} = \sin x$; *i.e.*, $EF = \sin x \cdot OE$.

But in rt. $\triangle OEP$, $\dfrac{OE}{OP \text{ or } 1} = \cos y$; $\therefore EF = \sin x \cos y$. (2)

Value of PG:

In rt. $\triangle PGE$, $\dfrac{PG}{PE} = \cos x$; *i.e.*, $PG = \cos x \cdot PE$.

But in rt. $\triangle OEP$, $\dfrac{PE}{OP \text{ or } 1} = \sin y$; $\therefore PG = \cos x \sin y$. (3)

Substituting (2) and (3) in (1), we obtain

$$\mathbf{\sin(x-y) = \sin x \cos y - \cos x \sin y.}$$

In a similar manner, using the same diagram, it can be proved that

$$\cos(x - y) = \cos x \cos y + \sin x \sin y.$$

(The proof is left to the student.)

The formulas for $\sin(x - y)$ and $\cos(x - y)$ can be proved geometrically, as above, for the cases when x and y have any values and $(x - y)$ terminates in any of the four quadrants. However, due attention must be paid to the algebraic signs of the functions of the angles involved.

Formula for tan $(x - y)$. For all values of x and y,

$$\tan(x - y) = \frac{\sin(x - y)}{\cos(x - y)}$$

$$= \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y}. \qquad (1)$$

The right-hand member of equation (1) can be expressed in terms of tangents if we divide both numerator and denominator by $\cos x \cos y$. Thus we obtain,

$$\tan(x - y) = \frac{\dfrac{\sin x \cos y}{\cos x \cos y} - \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} + \dfrac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\dfrac{\sin x}{\cos x} - \dfrac{\sin y}{\cos y}}{1 + \dfrac{\sin x}{\cos x} \cdot \dfrac{\sin y}{\cos y}}.$$

$$\therefore \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$$

Formula for cot $(x - y)$. For all values of x and y,

$$\cot(x - y) = \frac{\cos(x - y)}{\sin(x - y)}$$

$$= \frac{\cos x \cos y + \sin x \sin y}{\sin x \cos y - \cos x \sin y}. \qquad (1)$$

The right-hand member of equation (1) can be expressed in terms of cotangents if we divide both numerator and denominator by $\sin x \sin y$. Thus we obtain,

$$\cot (x - y) = \frac{\dfrac{\cos x \cos y}{\sin x \sin y} + \dfrac{\sin x \sin y}{\sin x \sin y}}{\dfrac{\sin x \cos y}{\sin x \sin y} - \dfrac{\cos x \sin y}{\sin x \sin y}}$$

$$= \frac{\dfrac{\cos x}{\sin x} \cdot \dfrac{\cos y}{\sin y} + 1}{\dfrac{\cos y}{\sin y} - \dfrac{\cos x}{\sin x}}.$$

$$\therefore \cot (x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}.$$

Remember

1. $\sin (x - y) = \sin x \cos y - \cos x \sin y.$
2. $\cos (x - y) = \cos x \cos y + \sin x \sin y.$
3. $\tan (x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}.$
4. $\cot (x - y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}.$

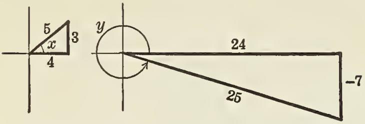
Illustrative examples.

Example 1. Find the value of $\cos 15°$ in radical form.

Solution

$$\begin{aligned} \cos 15° &= \cos (45° - 30°) \\ &= \cos 45° \cos 30° + \sin 45° \sin 30° \\ &= (\tfrac{1}{2}\sqrt{2})(\tfrac{1}{2}\sqrt{3}) + (\tfrac{1}{2}\sqrt{2})(\tfrac{1}{2}) \\ &= \tfrac{1}{4}\sqrt{6} + \tfrac{1}{4}\sqrt{2} \\ &= \tfrac{1}{4}(\sqrt{6} + \sqrt{2}). \end{aligned}$$

Example 2. If x is in Q I and $\sin x = \frac{3}{5}$ and y is in Q IV and $\tan y = -\frac{7}{24}$, find the value of $\tan(x - y)$.



Solution

$$\sin x = \frac{3}{5} = \frac{\text{ordinate}}{\text{distance}}.$$

$\therefore$ third side of basic triangle $= \sqrt{5^2 - 3^2} = 4$.

$$\tan y = -\tfrac{7}{24}.$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$= \frac{(\frac{3}{4}) - (-\frac{7}{24})}{1 + (\frac{3}{4})(-\frac{7}{24})} = \frac{\frac{3}{4} + \frac{7}{24}}{1 - \frac{21}{96}} = \frac{\frac{25}{24}}{\frac{75}{96}}.$$

$$\therefore \tan(x - y) = \tfrac{4}{3}.$$

Example 3. Solve $\sin(60° - x) - \sin(60° + x) = \dfrac{\sqrt{3}}{2}$ for all values of x between 0° and 360°.

Solution

$$\sin(60° - x) - \sin(60° + x) = \frac{\sqrt{3}}{2}.$$

$$(\sin 60° \cos x - \cos 60° \sin x) - (\sin 60° \cos x + \cos 60° \sin x) = \frac{\sqrt{3}}{2}.$$

$$\tfrac{1}{2}\sqrt{3}\cos x - \tfrac{1}{2}\sin x - \tfrac{1}{2}\sqrt{3}\cos x - \tfrac{1}{2}\sin x = \frac{\sqrt{3}}{2}.$$

$$-\sin x = \frac{\sqrt{3}}{2}.$$

$$\sin x = -\frac{\sqrt{3}}{2}.$$

[The angle whose sine is $\frac{1}{2}\sqrt{3}$ is 60°. But since $\sin x$ is $-$, one value of x is in Q III, $(180° + x)$, and another is in Q IV, $(360° - x)$.]

$$\therefore x_1 = 180° + 60° = 240°$$

and $$x_2 = 360° - 60° = 300°.$$

Show that both answers check in the original equation.

Example 4. Express $\cot y - \cot x$ in a form not containing sums or differences of functions.

Solution

$$\cot y - \cot x = \frac{\cos y}{\sin y} - \frac{\cos x}{\sin x}$$
$$= \frac{\sin x \cos y - \cos x \sin y}{\sin x \sin y}$$
$$= \frac{\sin (x - y)}{\sin x \sin y}.$$

EXERCISES

1. Expand each of the following:

(*a*) $\cos (45° - x)$. (*c*) $\tan (45° - \theta)$. (*e*) $\sin \left(A - \frac{2\pi}{3}\right)$.

(*b*) $\sin (x - 180°)$. (*d*) $\cot (\theta - 45°)$. (*f*) $\cos \left(\frac{5\pi}{6} - B\right)$.

2. Show that: (*a*) $\sin (45° - A) = \cos (45° + A)$.

(*b*) $\tan (45° - A) = \dfrac{1 - \tan A}{1 + \tan A}$.

(*c*) $\cos (60° - x) = \dfrac{\cos x + \sqrt{3} \sin x}{2}$.

(*d*) $\tan (45° + A) \tan (45° - A) = 1$.

3. Express in radical form:

(*a*) $\sin 15°$. (*b*) $\tan 15°$. (*c*) $\cot 15°$.

4. If $\sin x = \frac{12}{13}$ and x is acute, find the value of:

(*a*) $\sin (90° - x)$. (*c*) $\tan (45° - x)$.

(*b*) $\cos (60° - x)$. (*d*) $\cot (30° - x)$.

5. If $\sin x = \frac{12}{13}$, $\cos y = \frac{4}{5}$, and x and y are acute, find the value of:

(*a*) $\sin (x - y)$. (*c*) $\tan (x - y)$.

(*b*) $\cos (x - y)$. (*d*) $\cot (x - y)$.

6. If A and B are adjacent angles and $\tan A = \frac{2}{3}$ and $\tan B = \frac{3}{8}$, find the value of $\tan (A - B)$.

7. If x is in Q II and $\sin x = .6$ and y is in Q I and $\cos y = \frac{15}{17}$, find $\tan(x - y)$.

8. If x is in Q III and $\sin x = -\frac{5}{13}$, find the value of:

(a) $\sin(x - 90°)$.
(b) $\cos(270° - x)$.
(c) $\tan(x - 45°)$.
(d) $\cos(x - 60°)$.

Prove the following identities:

9. $\sin(30° - x) + \sin(30° + x) = \cos x$.

10. $\sin(45° + y) - \sin(45° - y) = \sqrt{2} \sin y$.

11. $\tan(60° + y) = \cot(30° - y)$.

12. $\sin(x + y) \sin(x - y) = \sin^2 x - \sin^2 y$.

13. $(\cos x + \cos y)^2 + (\sin x + \sin y)^2 = 2 + 2\cos(x - y)$.

14. $\cos(120° - A) + \cos(120° + A) = -\cos A$.

15. $\dfrac{\sin(A - B)}{\sin A \sin B} = \cot B - \cot A$.

16. $\cot x - \cot y = \dfrac{\sin(y - x)}{\sin x \sin y}$.

Solve the following equations for all values of x between 0° and 360°. Check.

17. $\tan(45° - x) + \cot(45° - x) = 4$.

18. $\sin(x - 30°) = \frac{1}{2}\sqrt{3} \sin x$.

19. $\tan\left(\frac{\pi}{4} + x\right) + \tan\left(\frac{\pi}{4} - x\right) = 4$.

20. $\sin(x - 30°) \sin(x + 30°) = \frac{1}{2}$.

21. $\sin\left(x + \frac{\pi}{4}\right) = 2\cos\left(x - \frac{\pi}{4}\right)$.

Express each of the following in a form not containing sums or differences of functions:

22. $\cot y + \cot x$.

23. $\cot x - \tan x$.

24. $\cot x + \tan x$.

25. $\cot x - \tan y$.

26. $\dfrac{\tan x + \tan y}{\cot y + \cot x}$.

Functions of Twice an Angle

In our study of trigonometry we frequently meet with expressions such as $\sin 2x$, $\cos 2y$, $\tan 2m$, and the like. If we substitute x for y in the addition formulas (page 346), we obtain formulas for the functions of a double angle. By means of these new formulas, the functions of a double angle can be reduced to the functions of a single angle.

Formula for sin 2 x.

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

Substituting x for y,

$$\sin(x+x) = \sin x \cos x + \cos x \sin x.$$

$$\therefore \boldsymbol{\sin 2x = 2\sin x \cos x.}$$

Formula for cos 2 x.

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

Substituting x for y,

$$\cos(x+x) = \cos x \cos x - \sin x \sin x.$$

$$\therefore \boldsymbol{\cos 2x = \cos^2 x - \sin^2 x.}$$

Formula for tan 2 x.

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

Substituting x for y,

$$\tan(x+x) = \frac{\tan x + \tan x}{1 - \tan x \tan x}.$$

$$\therefore \boldsymbol{\tan 2x = \frac{2\tan x}{1 - \tan^2 x}.}$$

Formula for cot 2 x.

$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}.$$

Substituting x for y,

$$\cot(x+x) = \frac{\cot x \cot x - 1}{\cot x + \cot x}.$$

$$\therefore \boldsymbol{\cot 2x = \frac{\cot^2 x - 1}{2\cot x}.}$$

Remember

1. $\sin 2x = 2 \sin x \cos x.$
2. $\cos 2x = \cos^2 x - \sin^2 x.$
3. $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}.$
4. $\cot 2x = \dfrac{\cot^2 x - 1}{2 \cot x}.$

Illustrative examples.

Example 1. Prove the identity $\sin 2x = \dfrac{2 \tan x}{1 + \tan^2 x}.$

Solution

$$\sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$$

$$\text{if} \quad 2 \sin x \cos x = \frac{2 \dfrac{\sin x}{\cos x}}{1 + \dfrac{\sin^2 x}{\cos^2 x}};$$

$$\text{if} \quad 2 \sin x \cos x = \frac{\dfrac{2 \sin x}{\cos x}}{\dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}};$$

$$\text{if} \quad 2 \sin x \cos x = \frac{2 \sin x}{\cos x} \times \frac{\cos^2 x}{1};$$

$$\text{if} \quad 2 \sin x \cos x = 2 \sin x \cos x.$$

But this is true, and since the steps are reversible,

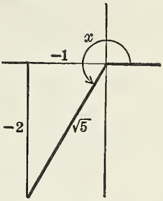
$$\sin 2x = \frac{2 \tan x}{1 + \tan^2 x}.$$

Example 2. Find the value of $\sin 3x$ in terms of $\sin x$.

Solution

$$\begin{aligned}
\sin 3x &= \sin (2x + x) \\
&= \sin 2x \cos x + \cos 2x \sin x \\
&= (2 \sin x \cos x) \cos x + (\cos^2 x - \sin^2 x) \sin x \\
&= 3 \sin x \cos^2 x - \sin^3 x \\
&= 3 \sin x (1 - \sin^2 x) - \sin^3 x. \\
\therefore \sin 3x &= 3 \sin x - 4 \sin^3 x.
\end{aligned}$$

Example 3. If x is in Q III and $\tan x = 2$, find the value of $\cos 2x$.



Solution

$$\tan x = \tfrac{2}{1}.$$

Since x is in Q III, the ordinate is $-$ and the abscissa is $-$.

$$\therefore \frac{\text{ordinate}}{\text{abscissa}} = \frac{-2}{-1}.$$

Third side of basic triangle $= \sqrt{(-2)^2 + (-1)^2} = \sqrt{5}$.

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= \left(\frac{-1}{\sqrt{5}}\right)^2 - \left(\frac{-2}{\sqrt{5}}\right)^2$$
$$= \tfrac{1}{5} - \tfrac{4}{5}.$$
$$\therefore \cos 2x = -\tfrac{3}{5}.$$

Example 4. Solve $\sin 2\theta = \frac{1}{2}\sqrt{3}$ for all values of θ between $0°$ and $360°$.

Solution

$$\sin 2\theta = \tfrac{1}{2}\sqrt{3}.$$

Letting $2\theta = x$, we obtain

$$\sin x = \tfrac{1}{2}\sqrt{3}.$$
$$\therefore x_1 = 60° \text{ and } x_2 = 120°.$$

Furthermore, other sets of answers may be obtained by adding 360°, 720°, etc., to the answers already found. Why?

$$\therefore x_3 = 60° + 360° = 420° \text{ and}$$
$$x_4 = 120° + 360° = 480°;$$
$$\text{i.e., } x = 60°, 120°, 420°, 480°, \text{etc.}$$

But since $2\theta = x$,

$$\theta = \frac{x}{2}.$$
$$\therefore \theta = 30°, 60°, 210°, 240°.$$

Verify each of the four answers by substitution in the original equation.

EXERCISES

1. If $\sin 30° = \frac{1}{2}$, find by use of the double angle formulas:
(a) $\sin 60°$. (b) $\cos 60°$. (c) $\tan 60°$. (d) $\cot 60°$.

2. If $\sin x = \frac{4}{5}$ and x is acute, find the value of:
(a) $\sin 2x$. (b) $\cos 2x$. (c) $\tan 2x$. (d) $\cot 2x$.

3. If $\cos y = -\frac{5}{13}$ and y is in Q II, find the value of:
(a) $\sin 2y$. (b) $\cos 2y$. (c) $\tan 2y$. (d) $\cot 2y$.

4. If $\tan \theta = \frac{1}{3}$ and θ is in Q III, find the value of:
(a) $\sin 2\theta$. (b) $\cos 2\theta$. (c) $\tan 2\theta$. (d) $\cot 2\theta$

5. (a) Given $\tan A = \frac{4}{3}$; find the value of $\cos 2A$.
(b) In which quadrant will $2A$ lie if A is in the first quadrant? in the third quadrant?

6. If $\cos x = -\frac{3}{5}$, x being in the second quadrant, find the value of $\cos 2x$. What, then, is the value of $\sec 2x$? In which quadrant does $2x$ lie?

Prove the following identities:

7. $(\sin A + \cos A)^2 = 1 + \sin 2A$.

8. $\cot A - \tan A = 2 \cot 2A$.

9. $\tan x + \cot x = 2 \csc 2x$.

10. $\csc 2y + \cot 2y = \cot y$.

11. $\cos^4 x - \sin^4 x = \cos 2x$.

12. $\dfrac{\sin 2x}{1 - \cos 2x} = \cot x$.

13. $\sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$.

14. $\dfrac{1 - \cos 2A}{\sin 2A} = \tan A$.

15. $\dfrac{1 - \cos 2A}{1 + \cos 2A} = \tan^2 A$.

16. $\dfrac{\sin 2x}{\sin x} - \dfrac{\cos 2x}{\cos x} = \sec x$.

Solve the following equations for all values of the unknown between 0° and 360°. Check.

17. $\sin 2x = 2 \cos x$.

18. $\cos 2x + \cos x = 0$.

19. $\cos 2x - 3\cos x - 1 = 0.$

20. $6 \sin x = 5 - 4 \cos 2x.$

21. $\cos 2x + \sin x = 1.$

22. $\tan y = \sin 2y.$

23. $2 \cos^3 \theta + \cos \theta \cos 2\theta = 0.$

24. $\cot y \tan 2y = 3.$

25. $\tan 2x - 3 \tan x = 0.$

26. $\dfrac{1}{\sec 2x} + \dfrac{2}{\sec x} = \dfrac{1}{2}.$

27. $\sin 2x = 0.$

28. $\sin 2x = 1.$

29. $2 \sin 2x = 1.$

30. $2 \sin 2x = \sqrt{2}.$

31. $2 \cos 2x = \sqrt{2}.$

32. $2 \cos 2x = \sqrt{3}.$

33. Express the value of $\tan 2x$ in terms of $\tan x$.

34. Express the value of $\sin 2x$ in terms of $\sin x$.

35. Express the value of $\cos 2x$ in terms of $\sin x$.

36. Express the value of $\cot 2x$ in terms of $\tan x$.

Functions of Half an Angle

In examples involving the functions of half angles, it is often best to change these functions to equivalent forms involving only the functions of whole angles. The new conversion formulas needed will now be derived.

Formula for $\sin \frac{1}{2} z$.

$$\cos^2 x + \sin^2 x = 1. \tag{1}$$

$$\cos^2 x - \sin^2 x = \cos 2x. \tag{2}$$

$$2 \sin^2 x = 1 - \cos 2x. \quad \text{[Subtract (2) from (1)]}$$

$$\therefore \sin^2 x = \frac{1 - \cos 2x}{2}.$$

$$\sin x = \pm \sqrt{\frac{1 - \cos 2x}{2}}. \tag{3}$$

If we let $2x = z$, then

$$x = \tfrac{1}{2} z. \tag{4}$$

Substituting (4) in (3), we obtain

$$\boldsymbol{\sin \tfrac{1}{2} z = \pm \sqrt{\frac{1 - \cos z}{2}}.}$$

Formula for $\cos \frac{1}{2} z$.

$$\cos^2 x + \sin^2 x = 1. \tag{1}$$

$$\cos^2 x - \sin^2 x = \cos 2x. \tag{2}$$

$$2\cos^2 x = 1 + \cos 2x. \quad [\text{Add (2) and (1)}]$$

$$\therefore \cos^2 x = \frac{1 + \cos 2x}{2}.$$

$$\cos x = \pm\sqrt{\frac{1 + \cos 2x}{2}}. \tag{3}$$

If we let $2x = z$, then $x = \frac{1}{2} z$. (4)

Substituting (4) in (3), we obtain

$$\cos \tfrac{1}{2} z = \pm\sqrt{\frac{1 + \cos z}{2}}.$$

Formula for $\tan \frac{1}{2} z$.

$$\tan \tfrac{1}{2} z = \frac{\sin \frac{1}{2} z}{\cos \frac{1}{2} z}.$$

[Substituting formulas for $\sin \frac{1}{2} z$ and $\cos \frac{1}{2} z$]

$$\tan \tfrac{1}{2} z = \frac{\pm\sqrt{\dfrac{1 - \cos z}{2}}}{\pm\sqrt{\dfrac{1 + \cos z}{2}}}.$$

$$\therefore \tan \tfrac{1}{2} z = \pm\sqrt{\frac{1 - \cos z}{1 + \cos z}}.$$

Formula for $\cot \frac{1}{2} z$.

$$\cot \tfrac{1}{2} z = \frac{\cos \frac{1}{2} z}{\sin \frac{1}{2} z}$$

$$= \frac{\pm\sqrt{\dfrac{1 + \cos z}{2}}}{\pm\sqrt{\dfrac{1 - \cos z}{2}}}.$$

$$\therefore \cot \tfrac{1}{2} z = \pm\sqrt{\frac{1 + \cos z}{1 - \cos z}}.$$

Something to think about.

1. If z is one of the angles of a triangle, should the double sign ($\pm$) be used in the half angle formulas just established?

2. If z is one of the angles of a quadrilateral, should the double sign ($\pm$) be used in the half angle formulas?

3. If z is a negative angle in Q III, is $\cos \frac{1}{2} z$ positive or negative?

4. Why is $(1 - \cos z)$ never negative? Why is $(1 + \cos z)$ never negative?

Remember

1. $\sin \frac{1}{2} z = \pm \sqrt{\dfrac{1 - \cos z}{2}}$.

2. $\cos \frac{1}{2} z = \pm \sqrt{\dfrac{1 + \cos z}{2}}$.

3. $\tan \frac{1}{2} z = \pm \sqrt{\dfrac{1 - \cos z}{1 + \cos z}}$.

4. $\cot \frac{1}{2} z = \pm \sqrt{\dfrac{1 + \cos z}{1 - \cos z}}$.

Illustrative examples.

Example 1. Prove the identity $\dfrac{1 + \sec z}{\sec z} = 2 \cos^2 \frac{1}{2} z$.

Solution
$$\frac{1 + \sec z}{\sec z} = 2 \cos^2 \tfrac{1}{2} z$$

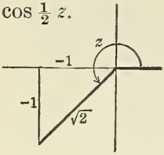
if
$$\frac{1 + \dfrac{1}{\cos z}}{\dfrac{1}{\cos z}} = 2\left[\pm \sqrt{\frac{1 + \cos z}{2}}\right]^2;$$

if
$$\frac{\cos z + 1}{\cancel{\cos z}} \cdot \frac{\cancel{\cos z}}{1} = \frac{\cancel{2}(1 + \cos z)}{\cancel{2}};$$

if
$$1 + \cos z = 1 + \cos z.$$

But this is true, and since the steps are reversible,
$$\frac{1 + \sec z}{\sec z} = 2 \cos^2 \tfrac{1}{2} z.$$

Example 2. If z is in Q III and $\cot z = 1$, find the value of $\cos \frac{1}{2} z$.



Solution

$$\cot z = \tfrac{1}{1}.$$

In Q III, the abscissa is $-$ and the ordinate is $-$.

$$\therefore \frac{\text{abscissa}}{\text{ordinate}} = \frac{-1}{-1}.$$

Third side of basic triangle $= \sqrt{2}$.

Since z is in Q III, the value of z is between 180° and 270°.

$\therefore$ the value of $\frac{1}{2} z$ is between 90° and 135° and $\frac{1}{2} z$ is in Q II and $\cos \frac{1}{2} z$ is $-$. Therefore only the $-$ sign should be used in the formula for $\cos \frac{1}{2} z$.

Hence $\cos \frac{1}{2} z = -\sqrt{\dfrac{1 + \cos z}{2}}$

$$= -\sqrt{\frac{1 + \left(-\dfrac{1}{\sqrt{2}}\right)}{2}} = -\sqrt{\frac{1 - \dfrac{\sqrt{2}}{2}}{2}} = -\sqrt{\frac{2 - \sqrt{2}}{4}}.$$

$$\therefore \cos \tfrac{1}{2} z = -\tfrac{1}{2}\sqrt{2 - \sqrt{2}}.$$

Example 3. Solve $\tan^2 \frac{1}{2} x + \cos x = \frac{5}{6}$ for all values of x between 0° and 360°.

Solution

$$\tan^2 \tfrac{1}{2} x + \cos x = \tfrac{5}{6}.$$

$$\left(\pm\sqrt{\frac{1 - \cos x}{1 + \cos x}}\right)^2 + \cos x = \tfrac{5}{6}.$$

$$\frac{1 - \cos x}{1 + \cos x} + \cos x = \tfrac{5}{6}.$$

$$6(1 - \cos x) + 6 \cos x\,(1 + \cos x) = 5(1 + \cos x).$$

$$6 \cos^2 x - 5 \cos x + 1 = 0.$$

$$(2 \cos x - 1)(3 \cos x - 1) = 0.$$

$\therefore \cos x = \frac{1}{2}$.	$\therefore \cos x = \frac{1}{3} = .3333$.
$\therefore x_1 = 60°$	$\therefore x_3 = 71°$ (to nearest degree)
and $x_2 = 360° - 60° = 300°$.	and $x_4 = 360° - 71° = 289°$.

$$\therefore x = 60°, 300°, 71°, 289°.$$

EXERCISES

1. Complete the table:

	If $\angle A$ is in	then $\angle 2A$ is in	and $\angle \frac{1}{2}A$ is in
(a)	Q I	Q I or Q II	Q I
(b)	Q II	?	?
(c)	Q III	?	?
(d)	Q IV	?	?

2. If $\cos 60° = \frac{1}{2}$, find, by use of the half-angle formulas:
(a) $\sin 30°$. (b) $\cos 30°$. (c) $\tan 30°$. (d) $\cot 30°$.

In each of the following examples, find (a) $\sin \frac{1}{2}x$, (b) $\cos \frac{1}{2}x$, (c) $\tan \frac{1}{2}x$, (d) $\cot \frac{1}{2}x$:

3. If $\cos x = \frac{3}{5}$ and x is acute.

4. If $\cos x = -\frac{5}{13}$ and x is in Q II.

5. If $\tan x = -\frac{5}{12}$ and x is in Q IV.

6. If $\sin x = -.8$ and x is in Q III.

7. If $\tan x = \frac{8}{15}$ and x is in Q III.

Prove the following identities:

8. $1 + \tan \dfrac{y}{2} \tan y = \sec y.$

9. $1 - \cos A = 2 \sin^2 \dfrac{A}{2}.$

10. $\tan \frac{1}{2}\theta = \csc \theta - \cot \theta.$

11. $\left(\cos \dfrac{A}{2} - \sin \dfrac{A}{2}\right)^2 = 1 - \sin A.$

12. $\tan \frac{1}{2}A = \dfrac{\sin A}{1 + \cos A}.$

13. $\cot \frac{1}{2}A = \dfrac{\sin A}{1 - \cos A}.$

14. $\sin z = \dfrac{2 \tan \frac{1}{2}z}{1 + \tan^2 \frac{1}{2}z}.$

15. $\tan z = \dfrac{2 \tan \frac{1}{2}z}{1 - \tan^2 \frac{1}{2}z}.$

Solve the following equations for all values of the unknown between 0° and 360°. Check.

16. $\cos \frac{1}{2} y = \sin y.$

17. $\tan \frac{1}{2} y = \sin y.$

18. $\dfrac{1}{\cot^2 \frac{1}{2} x} + \dfrac{1}{\sec x} = \dfrac{5}{6}.$

19. $\cot \frac{1}{2} \theta = 2 - \csc \theta.$

20. $\sin \frac{1}{2} \theta - \cos \frac{1}{2} \theta = \dfrac{\sqrt{2}}{2}.$

21. Express in terms of $\cos \frac{1}{2} \theta$ all the functions of θ.

22. Rationalize the right member of $\tan \frac{1}{2} \theta = \pm \sqrt{\dfrac{1 - \cos \theta}{1 + \cos \theta}}.$

Sums and Differences of Functions

Frequently in proving identities and solving equations, it is desirable to transform sums and differences of functions into equivalent products. Also, as you will learn later in the chapter on logarithms, such transformations are often helpful. We shall now derive the formulas which make possible transformations of this kind.

Formula for $\sin A + \sin B$.

$$\sin (x + y) = \sin x \cos y + \cos x \sin y. \qquad (1)$$

$$\sin (x - y) = \sin x \cos y - \cos x \sin y. \qquad (2)$$

$$\therefore \sin (x + y) + \sin (x - y) = 2 \sin x \cos y. \qquad (3)$$

Let $x + y = A$ and $x - y = B$.

Then $x = \frac{1}{2}(A + B)$ and $y = \frac{1}{2}(A - B). \qquad (4)$

Substituting (4) in (3), we obtain

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B).$$

Formula for $\sin A - \sin B$.

Subtract (2) from (1) above and show similarly that:

$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B).$$

Formula for cos A + cos B.

$$\cos(x+y) = \cos x \cos y - \sin x \sin y. \tag{1}$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y. \tag{2}$$
$$\therefore \cos(x+y) + \cos(x-y) = 2 \cos x \cos y. \tag{3}$$

Let $\quad x + y = A \quad$ and $\quad x - y = B.$

Then $\quad x = \frac{1}{2}(A + B) \quad$ and $\quad y = \frac{1}{2}(A - B). \qquad (4)$

Substituting (4) in (3), we obtain

$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B).$$

Formula for cos A − cos B.

Subtract (2) from (1) above and show similarly that:

$$\cos A - \cos B = -2 \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B)$$

Remember

1. $\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B).$
2. $\sin A - \sin B = 2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B).$
3. $\cos A + \cos B = 2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B).$
4. $\cos A - \cos B = -2 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B).$

Illustrative examples.

Example 1. Without using the tables prove that

$$\frac{\sin 75^\circ + \sin 15^\circ}{\sin 75^\circ - \sin 15^\circ} = \sqrt{3}.$$

Solution

Using the formulas for sin A + sin B and sin A − sin B,

$$\frac{\sin 75^\circ + \sin 15^\circ}{\sin 75^\circ - \sin 15^\circ} = \frac{\cancel{2} \sin \frac{1}{2}(75^\circ + 15^\circ) \cos \frac{1}{2}(75^\circ - 15^\circ)}{\cancel{2} \cos \frac{1}{2}(75^\circ + 15^\circ) \sin \frac{1}{2}(75^\circ - 15^\circ)}$$
$$= \frac{\sin 45^\circ \cos 30^\circ}{\cos 45^\circ \sin 30^\circ}$$
$$= \frac{\cancel{(\frac{1}{2}\sqrt{2})}(\frac{1}{\cancel{2}}\sqrt{3})}{\cancel{(\frac{1}{2}\sqrt{2})}(\frac{1}{\cancel{2}})}.$$
$$\therefore \frac{\sin 75^\circ + \sin 15^\circ}{\sin 75^\circ - \sin 15^\circ} = \sqrt{3}.$$

Example 2. Express as a product: $\cos 7x + \cos 5x$.

Solution

Using the formula for $\cos A + \cos B$,

$$\cos 7x + \cos 5x = 2\cos \tfrac{1}{2}(7x + 5x)\cos \tfrac{1}{2}(7x - 5x).$$

$$\therefore \cos 7x + \cos 5x = 2\cos 6x \cos x.$$

Example 3. Solve $\sin 5x - \sin 3x = \sqrt{2}\cos 4x$ for all values of x between 0° and 180°.

Solution

$$\begin{aligned} \sin 5x - \sin 3x &= \sqrt{2}\cos 4x. \\ 2\cos \tfrac{1}{2}(5x + 3x)\sin \tfrac{1}{2}(5x - 3x) &= \sqrt{2}\cos 4x. \\ 2\cos 4x \sin x &= \sqrt{2}\cos 4x. \\ 2\cos 4x\sin x - \sqrt{2}\cos 4x &= 0. \\ \cos 4x\,(2\sin x - \sqrt{2}) &= 0. \end{aligned}$$

$\cos 4x = 0.$	$2\sin x - \sqrt{2} = 0.$
$\therefore 4x = 90°, 270°, 450°, 630°$, etc.	$\therefore \sin x = \frac{1}{2}\sqrt{2}.$
$x = 22\frac{1}{2}°, 67\frac{1}{2}°, 112\frac{1}{2}°, 157\frac{1}{2}°,$	$\therefore x = 45°, 135°.$

$$\therefore x = 22\tfrac{1}{2}°, 45°, 67\tfrac{1}{2}°, 112\tfrac{1}{2}°, 135°, 157\tfrac{1}{2}°.$$

Check the first and second answers.

EXERCISES

Show by the sums to products formulas that each of the following is true:

1. $\sin 55° + \sin 5° = \sin 65°$.
2. $\cos 70° + \cos 50° = \cos 10°$.
3. $\cos 80° - \cos 20° = -\sin 50°$.
4. $\sin 80° + \sin 40° = \sqrt{3}\cos 20°$.
5. $\sin 70° - \sin 10° = \cos 40°$.
6. $\cos 11° - \cos 1° = -2\sin 6° \sin 5°$.

7. $\dfrac{\sin 36^\circ + \sin 6^\circ}{\cos 36^\circ + \cos 6^\circ} = \tan 21^\circ.$

8. $\dfrac{\sin 80^\circ + \sin 10^\circ}{\cos 80^\circ + \cos 10^\circ} = 1.$

9. $\dfrac{\sin 75^\circ + \sin 15^\circ}{\sin 75^\circ - \sin 15^\circ} = \sqrt{3}.$

10. $\cos \dfrac{\pi}{3} + \cos \dfrac{\pi}{2} = 2 \cos \dfrac{5\pi}{12} \cos \dfrac{\pi}{12}.$

11. $\dfrac{\sin 3\theta + \sin 2\theta}{\cos 3\theta - \cos 2\theta} = -\cot \dfrac{\theta}{2}.$

12. $\cos (30^\circ + A) - \cos (30^\circ - A) = -\sin A.$

13. $\sin \left(\dfrac{\pi}{4} + A\right) - \sin \left(\dfrac{\pi}{4} - A\right) = \sqrt{2} \sin A.$

14. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}.$

Express each of the following as a product:

15. $\sin 10x - \sin 6x.$

16. $\cos 7x + \cos 5x.$

17. $\sin 5A + \sin A.$

18. $\cos 2\theta - \cos \theta.$

19. $\sin 5x + \sin 11x.$

20. $\sin 5x - \sin 11x.$

21. $\sin \left(\dfrac{\pi}{9} + y\right) - \sin \left(\dfrac{\pi}{9} - y\right).$

22. $\dfrac{\sin A - \sin B}{\sin A + \sin B}.$

23. $\cos (\frac{2}{3}\pi + x) - \cos (\frac{2}{3}\pi - x).$

24. $\dfrac{\sin A + \sin B}{\cos A + \cos B}.$

Solve for all values of x between 0° and 360°. Check.

25. $\cos 3x + \cos x = \cos 2x.$

26. $\sin 5x - \sin 3x = -\sin x.$

27. $\cos 5x - \cos 3x = -\sin x.$

28. $\sin 2x + \sin x = \cos 2x + \cos x.$

29. $\sin 3x + \sin 2x + \sin x = 0.$

30. $\cos 3x + \cos 2x + \cos x = 0.$

MISCELLANEOUS EXERCISES

Group A

1. Using the functions of 60° and 45°, show that

$$\tan 15° = \frac{\sqrt{3} - 1}{1 + \sqrt{3}}.$$

2. If x is in Q II and $\sin x = \frac{24}{25}$, find the value of:
(*a*) $\sin (180° - x)$.
(*b*) $\sin (270° + x)$.
(*c*) $\cos (x - 90°)$.
(*d*) $\tan (x + 45°)$.
(*e*) $\tan 2x$.
(*f*) $\cot \frac{1}{2} x$.

3. If x is in Q I and $\sin x = \frac{3}{5}$, and y is in Q II and $\cos y = -\frac{5}{13}$, find the value of:
(*a*) $\sin (x + y)$.
(*b*) $\cos (x - y)$.
(*c*) $\tan \frac{1}{2} x$.
(*d*) $\cot 2y$.
(*e*) $\sin x + \sin y$.

4. Express as a product instead of a sum or difference:
(*a*) $\cos 10x + \cos 6x$.
(*b*) $\sin \frac{5x}{2} - \sin \frac{3x}{2}$.
(*c*) $\frac{\sin 5x + \sin 3x}{\sin 5x - \sin 3x}$.
(*d*) $\frac{\sin 3x - \sin 5x}{\sin 3x + \sin 5x}$.

5. Show that $\angle A$ will lie in either Q I or Q II if:
(*a*) $\tan \frac{1}{2} A$ is positive;
(*b*) $2A < 360°$.

Prove the following identities:

6. $\frac{\cos x}{1 - \sin x} = \tan\left(45° + \frac{x}{2}\right)$.

7. $\frac{\sin (x + y)}{\sin (x - y)} = \frac{\cot y + \cot x}{\cot y - \cot x}$.

8. $\cos 2B = \frac{1 - \tan^2 B}{1 + \tan^2 B}$.

9. $\frac{1 - \cos A}{1 + \cos A} = \tan^2 \frac{A}{2}$.

10. $\csc B - \cot B = \tan \frac{B}{2}$.

Solve for all values of the unknown between 0° and 360°. Check.

11. $\sin^2 2x = 2\cos^2 x$.

12. $\tan 2x - 2\tan x = 0$.

13. $\cos 2x = \sin x$.

14. $\cos 2x + \cos x + 1 = 0$.

15. $\sin\left(x + \frac{\pi}{3}\right) = \sin x$.

16. $\cos\left(y - \frac{\pi}{6}\right) = -\cos y$.

17. Express in terms of $\cos x$: (*a*) $\sin 2x$; (*b*) $\cos 2x$.

Group B

18. Using the functions of 45° and 30° in the expansion of $\cos(x + y)$, show that $\cos 75° = \dfrac{\sqrt{3} - 1}{2\sqrt{2}} = \dfrac{\sqrt{6} - \sqrt{2}}{4}$.

19. If $\tan x = -\dfrac{\sqrt{7}}{3}$, x being in Q IV; and $\cos y = -\frac{7}{25}$, y being in Q II, find the value of:

(*a*) $\tan(x + y)$.

(*b*) $\cos(x - y)$.

(*c*) $\cot \frac{1}{2} y$.

(*d*) $\tan 2x$.

(*e*) $\cos x + \cos y$.

20. Show that $\angle A$ may be in any one of the four quadrants if:

(*a*) $\sin \frac{1}{2} A$ is positive or negative.

(*b*) $\cos \frac{1}{2} A$ is positive or negative.

(*c*) $\tan 2A$ is positive and $2A$ is less than 720°.

21. Show that if $\tan \frac{1}{2} A$ is negative, $\angle A$ will lie in either Q III or Q IV.

22. Express the value of $\cos 3x$ in terms of $\cos x$.

23. Prove by a sums to products formula that:

(*a*) $\cos 88° + \cos 32° = \cos 28°$.

(*b*) $\sin 70° - \sin 10° = \sin 130°$.

24. Given: $\sin 52° = .7880$. $\cos 52° = .6157$.
$\sin 1° = .0175$. $\cos 1° = .9998$.

Find: (*a*) $\sin 53°$. (*c*) $\cos 53°$.
(*b*) $\sin 51°$. (*d*) $\cos 51°$.

25. Given: $\sin \frac{1}{2} x = \frac{3}{5}$, x being in Q II.
Find: (*a*) $\tan x$; (*b*) $\cos \frac{1}{2} x$; (*c*) $\sin 2x$.

26. Given: $\tan 2x = -\frac{4}{3}$, x being in Q I.
Find: (*a*) $\sin x$; (*b*) $\tan \frac{1}{2} x$; (*c*) $\cos 2x$.

Prove the following identities:

27. $\tan\left(x - \frac{\pi}{4}\right) + \cot\left(x + \frac{\pi}{4}\right) = 0.$

28. $\sin^4 x - \cos^4 x = -\cos 2x.$

29. $\csc(x + y) = \dfrac{\csc x \csc y}{\cot y + \cot x}.$

30. $\sec(x + y) = \dfrac{\sec x \sec y}{1 - \tan x \tan y}.$

31. $\tan\left(\frac{\pi}{4} + A\right) - \tan\left(\frac{\pi}{4} - A\right) = \dfrac{2}{\cot 2A}.$

Solve for all values of the unknown between 0° and 360°. Check.

32. $\cos 4x = \cos 2x.$

33. $\tan(45° + y) = \tan 2y + 2.$

34. $\sin\left(\frac{\pi}{3} - x\right) - \sin\left(\frac{\pi}{3} + x\right) = \dfrac{\sqrt{3}}{2}.$

35. $\tan\left(x + \frac{\pi}{4}\right) = 1 + \sin 2x.$

36. $\tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta - \frac{\pi}{3}\right) = 4.$

Group C

37. Without tables, find the value of:
(*a*) $\cos 105°$. (*b*) $\tan 7\frac{1}{2}°$. (*c*) $\sin 165°$.

38. Without the use of tables show that
$$\cos 20° \cos 40° \cos 80° = \tfrac{1}{8}.$$

39. Prove that: (*a*) $\cos 20° + \cos 100° + \cos 140° = 0.$
(*b*) $\sin 160° - \sin 100° + \sin 40° = 0.$

40. Show that $\tan \frac{1}{2} x$ and $\sin x$ always have the same sign.

41. Express the value of $\tan 4x$ in terms of $\tan x$.

42. Derive a formula for $\sec (x + y)$ in terms of secants and tangents.

43. Given: $\tan (x - y) = \frac{1}{2}$, $\tan y = \frac{1}{5}$, find $\tan x$.

Prove the following identities:

44. (*a*) $\tan^2 \theta = \dfrac{1 - \cos 2\theta}{1 + \cos 2\theta}$.

(*b*) Using (*a*) as a formula, find, without the use of tables, the value of $\tan 67\frac{1}{2}°$.

45. $\dfrac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B} = \tan (A + B) \tan (A - B)$.

46. $\dfrac{\sec x + \tan x}{\sec x - \tan x} = \dfrac{\tan \left(\frac{\pi}{4} + \frac{x}{2}\right)}{\tan \left(\frac{\pi}{4} - \frac{x}{2}\right)}$.

47. $\dfrac{\cos^3 x + \sin^3 x}{\cos x + \sin x} = \dfrac{2 - \sin 2x}{2}$.

48. $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$.

Solve for all values of the unknown between 0° and 360°. Check.

49. $\tan \left(y + \frac{\pi}{6}\right) = \cot y$.

50. $\sin 4x - 2 \sin 2x = 0$.

51. $\cos 3x + \sin 2x = \cos x$.

52. $\sin 3y + \sin y = 0$.

53. $\tan x + \tan 2x - \tan 3x = 0$.

54. Express the value of $\tan 3x$ in terms of $\tan x$.

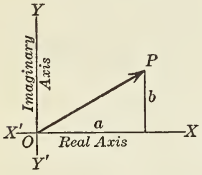
55. Multiply $\cos x + i \sin x$ by $\cos y + i \sin y$ and show that the result is equal to $\cos (x + y) + i \sin (x + y)$.

56. Prove that $(\cos x + i \sin x)^2 = \cos 2x + i \sin 2x$.

Graphic representation of the complex number.

Problem. A man is walking 6 feet a second across a boat that is traveling 8 feet a second. How fast is he actually traveling and in what direction?

This and many other problems in motion, force, and electricity are easily solved by means of the complex number $a + bi$. Before solving this problem it will be necessary to learn how complex numbers are represented graphically.



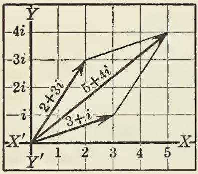
To represent $a + bi$ graphically, let XX' be the axis of real numbers and YY' of pure imaginaries. Then we can find a point P whose coördinates are (a, b) just as we did in plotting any point, by laying off distance a on the real axis XX' and distance b parallel to the imaginary axis YY', as shown in the adjoining figure. Then the point P (or the line OP, known as a *vector*) represents the complex number $a + bi$.

EXERCISES

Represent graphically the following complex numbers:

1. $2 + 3i$.	**4.** $-3 + 2i$.	**7.** $3 + i$.
2. $4 + i$.	**5.** $-5i - 2$.	**8.** $2 + 3i$.
3. $1 - 2i$.	**6.** $3\frac{1}{2} + 2\frac{1}{2}i$.	**9.** $5 + 4i$.

10. Notice that the number in Ex. 9 is the sum of those in Exs. 7 and 8. If all three are plotted on the same axes as in the adjoining diagram, what is the relation of $5 + 4i$ to the parallelogram whose sides are $3 + i$ and $2 + 3i$?



Graphic addition of complex numbers. In Ex. 7–10 above, we noted that the vector representing the sum of two complex numbers was the diagonal of the parallelogram whose sides were the vectors of the two numbers. Show by geometry that this relationship is true for any two complex numbers.

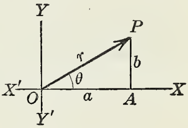
EXERCISES

Plot both numbers on the same axes and find their sum graphically. Check by finding the sum algebraically.

1. $4 + 2i$ and $1 + 4i$.
2. $3 + 3i$ and $3 - i$.
3. $-4+5i$ and $-2-5i$.
4. $2 + 6i$ and $2 - 6i$.
5. $3i + 4$ and $4i - 3$.
6. $-5-2\sqrt{-1}$ and $2\sqrt{-1}$.
7. $-3 + 3i$ and 3.
8. $1+\sqrt{-3}$ and $1-\sqrt{-3}$.
9. 8 and $6i$.
10. $4 + 2i$ and $-4 - 2i$.

Something to think about. How would you find graphically the difference between two complex numbers?

Polar representation of the complex number. Let OP represent any complex number $a + bi$; let the vector $OP = r$; and let $\theta = \angle XOP$.



Then $\dfrac{a}{r} = \cos\theta$ and $\dfrac{b}{r} = \sin\theta$.

$\therefore\ a = r\cos\theta$ and $b = r\sin\theta$.

Substituting these values of a and b in the form $a + bi$,

$$\begin{aligned} a + bi &= r\cos\theta + ri\sin\theta \\ &= r(\cos\theta + i\sin\theta). \end{aligned}$$

$$\boldsymbol{a + bi = r(\cos\theta + i\sin\theta).}$$

θ is called the *amplitude* and r the *modulus* of the complex number, or r and θ together are called the *polar coördinates* of the point P.

If a and b are known, we can find r and θ, for in the right triangle AOP

$$r = \sqrt{a^2 + b^2}$$

and $$\tan\theta = \frac{b}{a}.$$

Illustrative example. Express $3 + 4i$ in polar form.

Solution

Here $a = 3$ and $b = 4$.

$$\therefore r = \sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2} = 5.$$

$$\tan \theta = \frac{b}{a} = \frac{4}{3} = 1.3333.$$

$$\theta = 53^\circ \text{ (approximately).}$$

But $a + bi = r(\cos \theta + i \sin \theta)$.

$$\therefore 3 + 4i = 5(\cos 53^\circ + i \sin 53^\circ).$$

EXERCISES

Express each of the following complex numbers in polar form:

1. $4 + 3i$. **4.** $-3 + 4i$. **7.** $3i$.

2. $1 + i$. **5.** $-3i - 2$. **8.** -4.

3. $1 - i$. **6.** $-1 + 2i$. **9.** 2.

10. Show that the vectors of $-3 + 2i$ and $-2 - 3i$ are perpendicular.

Express each of the following in the form $a + bi$:

11. $\dfrac{3 + 2\sqrt{-1}}{2}$.

12. $\dfrac{-3 + \sqrt{-1}}{3}$.

13. $\dfrac{-4 + \sqrt{-4}}{2}$.

14. $5(\cos 30^\circ + i \sin 30^\circ)$.

15. $\sqrt{2}(\cos 135^\circ + i \sin 135^\circ)$.

16. $\cos 200^\circ - i \sin 200^\circ$.

17. $7(\cos 90^\circ + i \sin 90^\circ)$.

We shall now be able to solve problems by using our knowledge of complex numbers.

Illustrative examples.

Example 1. A man is walking 6 feet a second across a boat that is traveling 8 feet a second. How fast is he actually traveling and in what direction?

Solution

Since the motions are at right angles, we can represent one, say the 8 ft. a second, as a real number and the other as a pure imaginary. Then $8 + 6i$ is the sum of the motions.

Then
$$r = \sqrt{6^2 + 8^2} = 10.$$
$$\tan\theta = \tfrac{6}{8} = .7500.$$
$$\therefore\ \theta = 37^\circ \text{ (approximately).}$$

Therefore, the man is traveling 10 feet a second in a direction making an angle of 37° with the path of the boat.

Example 2. A force of 30 lb. acts at an angle of 20° with the x-axis and a force of 40 lb. at an angle of 50° with the x-axis. Find the amount and direction of the resultant force.

Solution

Expressed as complex numbers, the two forces are:

$$30(\cos 20^\circ + i \sin 20^\circ) = 30(.9397 + .3420\,i) = 28.1910 + 10.2600\,i.$$

and

$$40(\cos 50^\circ + i \sin 50^\circ) = 40(.6428 + .7660\,i) = 25.7120 + 30.6400\,i.$$

Adding:
$$\begin{array}{r} 28.1910 + 10.2600\,i \\ \underline{25.7120 + 30.6400\,i} \\ 53.9030 + 40.9000\,i \end{array}$$

$$\tan\theta = \frac{40.9000}{53.9030} = .7588. \qquad \therefore\ \theta = 37^\circ \text{ (approx.).}$$

$$\sin\theta = \frac{40.9000}{r}. \qquad \therefore\ r = \frac{40.9000}{.6018} = 67.9 = 68 \text{ (approx.).}$$

The resultant is a force of 68 pounds acting at an angle of 37° with the x-axis.

Another way of finding the resultant of two forces. The resultant can be found approximately by plotting the complex numbers, completing the parallelogram, and applying the Pythagorean theorem.

EXERCISES

Find the amount and direction of the resultant of the two forces:

1. 10 lb. acting toward the east and 10 lb. acting toward the north.

2. 30 lb. acting east and 40 lb. acting south.

3. 100 lb. acting east and 80 lb. acting northeast.

4. 50 lb. acting 30° west of north and 70 lb. acting west.

5. 100 lb. acting south and 50 lb. acting 60° east of south.

6. John hit a hockey ball 30° east of south with a force of 60 lb. at the same time that Robert hit it in a northeast direction with a force of 50 lb. Determine in what direction the ball will go.

7. Henry is swimming 2 miles an hour across a river that is flowing 5 miles an hour. What is his actual speed and in what direction is he swimming?

8. A stone is thrown horizontally from a cliff at the rate of 50 ft. a second. At the end of 2 seconds gravity has given it a downward velocity of 64 ft. a second. In what direction is it then moving and at what speed?

9. A boat is traveling 10 miles an hour in a stream that flows 4 miles an hour. Find the speed of the boat when it is traveling (*a*) upstream; (*b*) downstream; (*c*) directly across the stream; (*d*) at an angle of 60° with the direction of the current.

10. If the rain is falling vertically 50 ft. a second and a car is moving through it 30 ft. a second, what direction will the drops move on a side window of the car?

The product of two complex numbers. If we multiply $(\cos x + i \sin x)$ by $(\cos y + i \sin y)$, we get

$$\begin{aligned}&\cos x \cos y + i \sin x \cos y + i \cos x \sin y - \sin x \sin y \\ &= (\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \cos x \sin y) \\ &= \cos(x + y) + i \sin(x + y).\end{aligned}$$

$$\therefore\ (\cos x + i \sin x)(\cos y + i \sin y) = \cos(x + y) + i \sin(x + y). \quad (1)$$

Then to multiply $r_1 (\cos x_1 + i \sin x_1)$ by $r_2 (\cos x_2 + i \sin x_2)$, we can use equation (1) to obtain directly:

$$[r_1(\cos x_1 + i \sin x_1)][r_2(\cos x_2 + i \sin x_2)] = r_1 r_2[\cos(x_1 + x_2) + i \sin(x_1 + x_2)]. \quad (2)$$

The product of two complex numbers is a number whose modulus is the product of the moduli of the two complex numbers and whose amplitude is the sum of the amplitudes of these numbers.

Multiplication of complex numbers. We can develop a formula for the expansion of $[r(\cos x + i \sin x)]^n$.

If we let $y = x$ in (1), we have

$$(\cos x + i \sin x)^2 = \cos 2x + i \sin 2x. \quad (3)$$

If we let $y = 2x$ in (1), we have

$$(\cos x + i \sin x)(\cos 2x + i \sin 2x) = \cos 3x + i \sin 3x.$$

But from (2), we can substitute $(\cos x + i \sin x)^2$ for the second parenthesis, getting

$$(\cos x + i \sin x)^3 = \cos 3x + i \sin 3x. \quad (4)$$

Continuing this process, we can obtain the formula:

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx. \quad (5)$$

$$\therefore\ [r(\cos x + i \sin x)]^n = r^n(\cos nx + i \sin nx). \quad (6)$$

This relationship is called *De Moivre's theorem.*

Now if we let $n = \frac{1}{m}$ in formula 6, we obtain

$$[r(\cos x + i \sin x)]^{\frac{1}{m}} = r^{\frac{1}{m}}\left(\cos \frac{x}{m} + i \sin \frac{x}{m}\right)$$

which we can use as a formula for finding the roots of a complex number.

The N Nth Roots of Unity

If you were asked for the fifth root of 1, you would promptly give 1 as your answer. You will undoubtedly be surprised to learn that there are four other entirely different answers to this question, and that these answers can be obtained readily by the use of De Moivre's theorem. In order that you may understand how these roots are obtained, it will be necessary for you to answer the following questions.

1. Find all the roots of each of the following equations:

(a) $x - 1 = 0$. (c) $x^3 - 1 = 0$.

(b) $x^2 - 1 = 0$. (d) $x^4 - 1 = 0$.

2. How many roots has equation (a)? (b)? (c)? (d)? How does the number of roots compare with the degree of the equation?

3. If the number of roots of an equation equals the degree of that equation, how many roots has $x^2 - 1 = 0$? $x^3 - 1 = 0$? $x^4 - 1 = 0$? $x^n - 1 = 0$?

4. How many values must the square root of 1 have? the cube root of 1? the fourth root of 1? the nth root of 1?

5. Write each of the roots obtained in 1 in the $a + bi$ form.

6. Represent these roots graphically on the same set of axes.

7. What do you observe concerning the location of all these roots?

8. If the modulus of a complex number is 1, what is the modulus of its square? of its cube? of its fourth power? of its nth power?

9. If the amplitude of a complex number is x, what is the amplitude of its square? of its cube? of its nth power?

10. If we multiply the amplitude of a number by 3 to obtain the amplitude of its cube, by what number should we multiply the amplitude of a number to obtain that of its cube root? of its fourth root? of its nth root?

11. Express each of the following complex numbers in its simplest form:

(a) $\cos 0° + i \sin 0°$.
(b) $\cos 90° + i \sin 90°$.
(c) $\cos 180° + i \sin 180°$.
(d) $\cos 270° + i \sin 270°$.
(e) $\cos 360° + i \sin 360°$.
(f) $\cos 720° + i \sin 720°$.
(g) $\cos 1080° + i \sin 1080°$.

The answers to these questions lead to the following conclusions:

1. Unity has n nth roots.

2. Unity can be expressed as $\cos 0° + i \sin 0°$, $\cos 360° + i \sin 360°$, $\cos 720° + i \sin 720°$, etc.

3. Since the length of the modulus of unity is 1, the modulus of its nth root will be 1.

4. All points at a distance 1 from the origin lie on a circle whose center is the origin and whose radius is 1.

5. The amplitude of the nth root of a number is found by dividing the amplitude of the number by n.

From these conclusions it is evident that the n nth roots of unity are represented by points on a circle whose center is the origin and whose radius is 1. They can therefore be found by the formula

$$[r(\cos x + i \sin x)]^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos \frac{x}{n} + i \sin \frac{x}{n}\right),$$

since the left-hand member equals 1 when $r = 1$ and $x = 0°$, 360°, 720°, etc. Therefore the n nth roots of 1 are:

$$\cos \frac{0°}{n} + i \sin \frac{0°}{n};\ \cos \frac{360°}{n} + i \sin \frac{360°}{n};\ \cos \frac{720°}{n} + i \sin \frac{720°}{n}, \text{ etc.}$$

Notice that these amplitudes are respectively $\frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}$, etc., of the angular distance around a point. That is, the circle is divided into n equal arcs.

The n nth roots of unity are the vertices of a regular polygon inscribed in a circle whose radius is 1 and whose center is at the origin.

Illustrative example. Find the 5 fifth roots of 1.

Solution

$\cos \frac{0°}{5} + i \sin \frac{0°}{5} = \cos 0° + i \sin 0° = 1.$
$\cos \frac{360°}{5} + i \sin \frac{360°}{5} = \cos 72° + i \sin 72° = .3090 + .9511\, i.$
$\cos \frac{720°}{5} + i \sin \frac{720°}{5} = \cos 144° + i \sin 144° = -.8090 + .5878\, i.$
$\cos \frac{1080°}{5} + i \sin \frac{1080°}{5} = \cos 216° + i \sin 216° = -.8090 - .5878\, i.$
$\cos \frac{1440°}{5} + i \sin \frac{1440°}{5} = \cos 288° + i \sin 288° = .3090 - .9511\, i.$

Notice that there are no other solutions, for if we take the next multiple of 360° (1800°) and divide by 5, we obtain 360°, which is our starting point.

EXERCISES

1. Find the product of:

(*a*) $2(\cos 40° + i \sin 40°)$ and $5(\cos 50° + i \sin 50°)$.
(*b*) $3(\cos 15° + i \sin 15°)$ and $4(\cos 20° + i \sin 20°)$.

2. Find the square of:

(*a*) $2(\cos 40° + i \sin 40°)$. (*c*) $\cos 120° + i \sin 120°$.
(*b*) $\sqrt{2}(\cos 225° + i \sin 225°)$. (*d*) $\cos 240° + i \sin 240°$.

(*e*) What peculiar fact do you notice about the squares of the numbers in (*c*) and (*d*)? Without using the polar form, square $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$ and then square your answer. Does the same peculiar relation appear?

3. Find the cube of:

(*a*) $\cos 60° + i \sin 60°$. (*c*) $\cos 120° + i \sin 120°$.
(*b*) $3(\cos 20° + i \sin 20°)$. (*d*) $\cos 240° + i \sin 240°$.

4. Find the 3 cube roots of 1; the 6 sixth roots of 1.

5. Find the square root of:

(*a*) i. (*b*) $-i$. (*c*) $\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{-2}$.

Inverse Trigonometric Functions

You have already seen that the values of the six trigonometric ratios of an angle depend upon the size of the angle. Because of this dependence the six trigonometric ratios are regarded as functions of the angle. Conversely, the size of the angle depends upon the value of any one of the six ratios. For example, the size of the angle depends upon the value of the sine. From this point of view, the angle may be said to be a function of the sine. Similarly the angle is a function of each of the other five ratios. The latter functions are called the *inverse* or *anti functions* of the former.

This inverse relationship as applied to the sine, for example, can be expressed as follows:

Direct	*Inverse*
The sine of the angle θ is m.	θ is the angle whose sine is m.

Or as usually stated:

$\sin\theta = m.$	$\theta = \sin^{-1} m.$

The symbol, $\sin^{-1} m$, is read "the angle whose sine is m," "the arc whose sine is m," "the anti-sine of m" or "the inverse sine of m."

Since the first of these statements is the most meaningful, it is preferred to the others.

Consider the expressions: (1) $\sin 30° = \frac{1}{2}$. (2) $x = \sin^{-1}\frac{1}{2}$. Expression (1) tells us that for the definite angle 30°, the value of the sine is $\frac{1}{2}$. Expression (2) asks what angle has a sine equal to $\frac{1}{2}$. Evidently the angle 30° answers this question. But 150° is also an answer since $\sin 150° = \frac{1}{2}$. But you have also learned that if 30° and 150° are answers to a trigonometric equation, other sets of answers are $360° + 30°$, $360° + 150°$, $720° + 30°$, $720° + 150°$; etc.

From the discussion we may say: *The trigonometric functions are single-valued, while the inverse trigonometric functions are many-valued.*

Of all the values of an inverse trigonometric function, the smallest positive value is called its *principal value.* Thus, in the example above, 30° is the principal value of $\sin^{-1}\frac{1}{2}$. Likewise the principal value of $\tan^{-1}(-1)$ is 135°. (Unless otherwise indicated, only the principal value of an inverse function will be considered.)

Illustrative examples.

Example 1. If $A = \sin^{-1}\frac{\sqrt{3}}{2}$, (1) state in which quadrants angle A may lie; (2) give the principal value of angle A in degrees and also in radians; (3) write the other value, less than 360°, which angle A may have; (4) construct the principal value of angle A; and (5) give the value of $\tan\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)$.

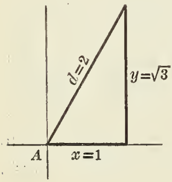
Solution

(1) Since A is the angle whose sine is $+\frac{\sqrt{3}}{2}$, angle A lies in either Q I or Q II.

(2) The principal value of angle A is 60° or $\frac{\pi}{3}$ radians.

(3) The other value of angle A is $180° - 60° = 120°$.

(4) If $$A = \sin^{-1}\frac{\sqrt{3}}{2},$$



then $$\sin A = \frac{\sqrt{3}}{2};$$

i.e., $$\frac{y \text{ (ordinate)}}{d \text{ (distance)}} = \frac{\sqrt{3}}{2}.$$

Since $$x^2 + y^2 = d^2,$$
$$x^2 + 3 = 4.$$
$$\therefore x = 1.$$

(5) From the above diagram, observe that:

$$\tan\left(\sin^{-1}\frac{\sqrt{3}}{2}\right) = \tan A = \sqrt{3}.$$

Example 2. Prove the identity

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy}.$$

Solution

Let $A = \tan^{-1} x$, $B = \tan^{-1} y$, and $C = \tan^{-1} \dfrac{x + y}{1 - xy}$.

Then $\tan A = x$, $\tan B = y$, and $\tan C = \dfrac{x + y}{1 - xy}$.

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy}$$

may now be written

$$A + B = C.$$

Since this is an expression of the equality of angles, any trigonometric function of the left member must equal the same function of the right member.

$$\therefore \tan (A + B) = \tan C.$$

Then
$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \tan C.$$

Substituting the values for $\tan A$, $\tan B$, and $\tan C$, we get

$$\frac{x + y}{1 - xy} = \frac{x + y}{1 - xy}.$$

Since these two members are identical, then the identity

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy} \text{ is true.}$$

Example 3. Solve the equation $\tan^{-1} 2y + \tan^{-1} 3y = 45°$.

Solution

Let $A = \tan^{-1} 2y$ and $B = \tan^{-1} 3y$.

Then $\tan A = 2y$ and $\tan B = 3y$.

The equation $\tan^{-1} 2y + \tan^{-1} 3y = 45°$ may now be written

$$A + B = 45°.$$

Since this is an expression of the equality of angles, any trigonometric function of the left member must equal the same function of the right member. Here it is best to use the tangent ratio.

$$\therefore \tan(A + B) = \tan 45^\circ \quad \text{or} \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1.$$

Substituting the values for $\tan A$ and $\tan B$, we get

$$\frac{2y + 3y}{1 - (2y)(3y)} = 1.$$

$$\frac{5y}{1 - 6y^2} = 1.$$

$$5y = 1 - 6y^2.$$

$$6y^2 + 5y - 1 = 0.$$

$$(6y - 1)(y + 1) = 0.$$

$$6y - 1 = 0. \qquad y + 1 = 0.$$

$$y = \tfrac{1}{6}. \qquad y = -1.$$

Check

$$y = \tfrac{1}{6}.$$

$$\tan^{-1} 2y + \tan^{-1} 3y = 45^\circ.$$

$$\tan^{-1} \tfrac{1}{3} + \tan^{-1} \tfrac{1}{2} = 45^\circ.$$

$$\tan^{-1} .3333 + \tan^{-1} .5000 = 45^\circ.$$

$$18^\circ + 27^\circ = 45^\circ.$$

$$45^\circ = 45^\circ.$$

Check

$$y = -1.$$

$$\tan^{-1} 2y + \tan^{-1} 3y = 45^\circ.$$

$$\tan^{-1}(-2) + \tan^{-1}(-3) = 45^\circ.$$

Using principal values only:

$117^\circ + 108^\circ$ does not equal 45°.

$\therefore y = -1$ must be rejected.

Example 4. Without using the tables solve

$$6 \sin^2 x - 5 \sin x + 1 = 0$$

for values of x between 0° and 360°.

Solution

$$6 \sin^2 x - 5 \sin x + 1 = 0.$$

$$(2 \sin x - 1)(3 \sin x - 1) = 0.$$

$$2 \sin x - 1 = 0. \qquad 3 \sin x - 1 = 0.$$

$$\sin x = \tfrac{1}{2}. \qquad \sin x = \tfrac{1}{3}.$$

$$\therefore x_1 = 30^\circ. \qquad \therefore x_3 = \sin^{-1} \tfrac{1}{3}.$$

$$x_2 = 180^\circ - 30^\circ = 150^\circ.$$

$$x = 30^\circ, 150^\circ, \sin^{-1} \tfrac{1}{3}.$$

EXERCISES

For each of the following examples (1) state in which quadrants angle A may lie; (2) give the principal value of angle A in degrees and also in radians; (3) write the other value, less than 360°, which angle A may have; and (4) construct the principal value of angle A.

1. $A = \sin^{-1} \frac{1}{2}$.
2. $A = \cos^{-1} \frac{\sqrt{3}}{2}$.
3. $A = \sec^{-1} 2$.
4. $A = \csc^{-1} \sqrt{2}$.
5. $A = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.
6. $A = \tan^{-1}(-1)$.
7. $A = \cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)$.
8. $A = \tan^{-1} 2$.

Give the value of each of the following:

9. $\sin(\cos^{-1} \frac{1}{2})$.
10. $\tan\left(\sin^{-1} \frac{\sqrt{3}}{2}\right)$.
11. $\cos(\tan^{-1} 1)$.
12. $\sin(\tan^{-1} 2)$.
13. $\sec(\tan^{-1} \frac{5}{12})$.
14. $\csc\left(\tan^{-1} \frac{2mn}{m^2 - n^2}\right)$.
15. $\cos(2 \tan^{-1} y)$.
16. $\tan(2 \cot^{-1} 1)$.
17. $\cot\left(2 \sin^{-1} \frac{1}{x}\right)$.
18. $\cos(\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{2})$.

Prove the following:

19. $\sin(2 \cos^{-1} \frac{3}{5}) = \frac{24}{25}$.
20. $\cos(2 \tan^{-1} y) = \frac{1 - y^2}{1 + y^2}$.
21. $\sin^{-1}(-y) = -\sin^{-1} y$.
22. $2 \sin^{-1} x = \cos^{-1}(1 - 2x^2)$.
23. $2 \tan^{-1} \frac{2}{3} - \tan^{-1} \frac{12}{5} = 0$.
24. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.
25. $\tan(2 \tan^{-1} x) = \frac{2x}{1 - x^2}$.
26. $\sin^{-1} a + \sin^{-1} \sqrt{1 - a^2} = \frac{\pi}{2}$.

27. $\tan(\tan^{-1} x + \tan^{-1} y) = \dfrac{x + y}{1 - xy}$.

28. $\sin^{-1} \frac{1}{2} + \sin^{-1} \frac{1}{2}\sqrt{3} = \dfrac{\pi}{2}$.

29. $\tan^{-1} \dfrac{1}{7} + 2 \tan^{-1} \dfrac{1}{3} = \dfrac{\pi}{4}$.

30. $\tan^{-1} \frac{1}{4} + \tan^{-1} \frac{1}{13} = \tan^{-1} \frac{1}{3}$.

31. $\sin^{-1}(\tan 45°) = 2 \sin^{-1} \dfrac{\sqrt{2}}{2}$.

Solve the following equations. Check.

32. $\sin^{-1} y = 2 \cos^{-1} y$.

33. $\sin^{-1} y = 2 \tan^{-1} y$.

34. $\tan^{-1} y + \tan^{-1} 2y = \tan^{-1} 3$.

35. $\tan^{-1} y = \tan^{-1} 7 - \tan^{-1} 2$.

36. $\cos^{-1} x + 2 \sin^{-1} x = \dfrac{\pi}{3}$.

37. $\tan^{-1} 3x + 2 \cot^{-1} x = 180°$.

Without using the tables solve each of the following equations for all values of the unknown from 0° to 360°. Express part of your answer as an inverse function.

38. $6 \sin^2 x + 5 \sin x + 1 = 0$.

39. $\tan^2 y - 3 \tan y + 2 = 0$.

40. $3 \cos^2 x + 2 = 5 \cos x$.

41. $5 \sin^2 y = 4 \sin y$.

42. $(2 \cos x - 1)(2 \sin x + \sqrt{3})(2 \tan x - 3) = 0$.

To the Stars Via Trigonometry

The "Almagest," the great work of Ptolemy (about 150 A.D.), contains a great deal of information on trigonometry. Being an astronomer, his trigonometry was useful only in so far as it aided him in his astronomical studies, and therefore the relations or formulas he derived were introduced solely for that purpose.

These formulas were improved upon and new ones added by Al-Battani (about 920 A.D.).

CUMULATIVE REVIEW

Chapters IX, X, and XI

1. Which of these statements are true? Which are false?

(*a*) $3x + y = 5$ and $x + 3y = 5$ are a pair of inconsistent equations.

(*b*) If $A = \sin^{-1} \frac{1}{3}$, the principal value of angle A is less than 45°.

(*c*) $2\sqrt{\frac{2}{3}} = \frac{2}{3}\sqrt{2}$.

(*d*) $2\sqrt[3]{-2} \times 3\sqrt[3]{-\frac{1}{2}} = 6$.

(*e*) $\dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$.

(*f*) $\sin(x + y) = \sin x + \sin y$.

(*g*) The formula, $\cos(x - y) = \cos x \cos y + \sin x \sin y$, does not hold when $y > x$.

(*h*) $\tan(45° - x) = \dfrac{1 - \tan x}{1 + \tan x}$.

(*i*) If $\sin 2B = -1$, then $B = 135°$.

(*j*) If angle B is obtuse and angle C is a right angle, then $\tan \frac{1}{2}(B + C)$ is positive.

2. Complete each of the following statements:

(*a*) If the graph of the equation $x - y = 1$ intersects the graph of the equation $x^2 + y^2 = 9$, the two equations have ? solutions. (Give number of solutions.)

(*b*) The obtuse angle which will satisfy the equations: $\cos x = y$ and $y = -\frac{1}{2}$ is ? .

(*c*) Expressed in terms of i: $4\sqrt{-4} =$? .

(*d*) The sum of: $\sqrt{-12} + 3\sqrt{-3}$ is ? .

(*e*) $\sqrt{-20} \times \sqrt{-5} =$? .

(*f*) Expressed in terms of $\tan x$, $\tan(45° + x) =$? .

(*g*) If $\tan A = 3$ and $\tan B = 2$, then $\tan(A - B) =$? .

(*h*) If A is an angle of a triangle such that $\cos A = -.8$, then the value of $\cos \frac{1}{2} A$ is ? .

(*i*) If $\sin x$ is negative and $\tan x$ is positive, angle $\frac{1}{2}x$ lies in quadrant ? .

(*j*) If $\cot z = -\sqrt{3}$, then the value of $\cot 2z$ is ? .

3. How many pounds of coffee at 50¢ a pound must be mixed with 100 pounds of coffee at 35¢ a pound to make a blend worth 42¢ a pound?

4. Solve for a and b, correctly group your answers, and check:

$$2a^2 + b^2 = 17$$
$$3b^2 - a^2 = 23.$$

5. Rationalize the denominator $\dfrac{8 - 2\sqrt{5}}{3 + \sqrt{5}}$ and find the value of the result correct to the nearest hundredth.

6. Solve for c: $2K = a\sqrt{c^2 - a^2}$.

7. Prove the identity:

$$\tan A = \sqrt{\frac{\tan A - \sin A \cos A}{\sin A \cos A}}.$$

8. Show that for all values of y:

$$\sin\left(\frac{\pi}{6} + y\right) = \cos\left(\frac{\pi}{3} - y\right).$$

9. If $\sin A = \frac{5}{13}$, $\cos B = \frac{3}{5}$ and A and B are acute angles, find the value of:

(*a*) $\tan(A + B)$.
(*b*) $\cot(A - B)$.
(*c*) $\cos 2B$.
(*d*) $\sin \frac{1}{2}A$.

10. Prove: $\sin 2A(\sec A + \csc A) = 2(\sin A + \cos A)$.

11. Solve for all values of θ between 0° and 360° and check the largest answer obtained: $\tan 2\theta = -2\sin\theta$.

12. Express $\tan 2x$ in terms of $\cot x$.

CHAPTER XII. THE MEANING AND USE OF EXPONENTS

Mathematics is the most marvelous instrument created by the genius of man for the discovery of truth. — LAISANT.

FRACTIONAL, ZERO, AND NEGATIVE EXPONENTS

You have already had considerable work with exponents. However, the exponents you have used were positive integers such as the ones in x^5, m^4, and b^2. You have also learned that $x^4 \div x^2 = x^{4-2} = x^2$.

Let us see how dividing monomials leads to peculiar results.

$$x^4 \div x^4 = x^{4-4} = x^0.$$

$$x^4 \div x^5 = x^{4-5} = x^{-1}.$$

Again, $\sqrt{a^8} = a^4$; $\sqrt{a^4} = a^2$; $\sqrt{a^2} = a$. Since in each case the exponent in the answer is one-half that in the previous answer, it seems to suggest that $\sqrt{a} = a^{\frac{1}{2}}$.

The question now is to determine whether these new exponents 0, -1, $\frac{1}{2}$, have any meaning. Before we try to discover any possible meanings, we shall review the laws of exponents.

Recall fact 54. *Law of exponents in multiplication:*

$$x^m \cdot x^n = x^{m+n}.$$

Thus $\qquad x^2 \cdot x^3 = x^5;\ x^5 \cdot x = x^6.$

Recall fact 55. *Law of exponents in division:*

$$x^m \div x^n = x^{m-n}.$$

Thus $\qquad x^5 \div x^2 = x^3;\ x^8 \div x = x^7.$

Recall fact 56. *Law of exponents in raising to powers:*

(*a*) $$(x^m)^n = x^{mn}.$$

Thus $$(x^5)^2 = x^{10}; \ (x^2)^5 = x^{10}.$$

(*b*) $$(x^a y^b)^m = x^{am} y^{bm}.$$

Thus $$(x^2y^3)^5 = x^{10}y^{15}; \ (xy^2)^3 = x^3y^6.$$

(*c*) $$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}.$$

Thus $$\left(\frac{x^2}{y^2}\right)^3 = \frac{x^6}{y^6}; \ \left(\frac{x^a}{y^b}\right)^n = \frac{x^{an}}{y^{bn}}.$$

The meaning of a fractional exponent. According to the law,

$$x^m \cdot x^n = x^{m+n},$$

$$x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x.$$

Since $x^{\frac{1}{2}}$ is one of the two equal factors of x, it is $\sqrt{x}$.

$$\therefore \ x^{\frac{1}{2}} = \sqrt{x}.$$

Similarly $x^{\frac{2}{3}} = \sqrt[3]{x^2}$ since $x^{\frac{2}{3}} \cdot x^{\frac{2}{3}} \cdot x^{\frac{2}{3}} = x^2$.

Again $x^{\frac{3}{4}} = \sqrt[4]{x^3}$ since $x^{\frac{3}{4}} \cdot x^{\frac{3}{4}} \cdot x^{\frac{3}{4}} \cdot x^{\frac{3}{4}} = x^3$.

In general $$x^{\frac{m}{n}} = \sqrt[n]{x^m}.$$

In other words: *When any quantity has a fractional exponent, the numerator of the exponent indicates the power to which the quantity is to be raised and the denominator indicates the root to be found.*

EXERCISES

Write the following as radicals:

1. $a^{\frac{2}{3}}$.	**6.** $y^{\frac{3}{8}}$.	**11.** $2a^{\frac{4}{9}}$.	**16.** $5n^{\frac{4}{5}}$.
2. $a^{\frac{4}{7}}$.	**7.** $b^{\frac{3}{2}}$.	**12.** $5x^{\frac{3}{7}}$.	**17.** $(6r)^{\frac{2}{3}}$.
3. $x^{\frac{1}{2}}$.	**8.** $x^{\frac{3}{4}}$.	**13.** $7a^{\frac{3}{5}}$.	**18.** $3x^{\frac{3}{4}}$.
4. $m^{\frac{4}{3}}$.	**9.** $m^{\frac{5}{6}}$.	**14.** $(8y)^{\frac{1}{2}}$.	**19.** $5x^{\frac{5}{2}}$.
5. $x^{\frac{1}{4}}$.	**10.** $x^{\frac{2}{5}}$.	**15.** $(2m)^{\frac{3}{2}}$.	**20.** $10y^{\frac{7}{8}}$.

Express each with a fractional exponent and simplify:

21. $\sqrt[4]{a^3}$.
22. $\sqrt[3]{a^4}$.
23. $\sqrt{y^3}$.
24. $\sqrt[3]{m^2}$.
25. $\sqrt[5]{m^{10}}$.
26. $\sqrt[4]{3x}$.
27. $\sqrt[3]{r^8}$.
28. $\sqrt[3]{m^2n^2}$.
29. $\sqrt{2a^3b^6}$.
30. $\sqrt[5]{ab^2}$.
31. $\sqrt{a^5b^3}$.
32. $\sqrt[6]{m^4b^6}$.
33. $\sqrt[3]{a^2b^4c^5}$.
34. $\sqrt{2a^2b^3x^{14}}$.
35. $2\sqrt[3]{m^2}$.
36. $4\sqrt[4]{m^3}$.
37. $5\sqrt{a}$.
38. $\sqrt{5a}$.
39. $3\sqrt[6]{m^5n}$.
40. $4\sqrt[7]{a^{21}b^3}$.
41. $\sqrt[3]{a^7b^2c}$.

Illustrative example. Find the value of $32^{\frac{4}{5}}$.

Solution

$$32^{\frac{4}{5}} = \sqrt[5]{32^4} = (\sqrt[5]{32})^4 = 2^4 = 16.$$

Find the value of each of the following:

42. $8^{\frac{1}{3}}$.
43. $9^{\frac{3}{2}}$.
44. $4^{\frac{1}{2}}$.
45. $9^{\frac{1}{2}}$.
46. $27^{\frac{1}{3}}$.
47. $64^{\frac{1}{3}}$.
48. $4^{\frac{3}{2}}$.
49. $25^{\frac{3}{2}}$.
50. $64^{\frac{2}{3}}$.
51. $27^{\frac{2}{3}}$.
52. $32^{\frac{1}{5}}$.
53. $32^{\frac{2}{5}}$.
54. $1^{\frac{3}{4}}$.
55. $(-8)^{\frac{2}{3}}$.
56. $(-27)^{\frac{2}{3}}$.
57. $(-32)^{\frac{1}{5}}$.
58. $8^{\frac{4}{3}}$.
59. $-27^{\frac{2}{3}}$.
60. $.027^{\frac{1}{3}}$.
61. $.04^{\frac{1}{2}}$.
62. $(\frac{1}{4})^{\frac{1}{2}}$.
63. $(\frac{1}{9})^{\frac{1}{2}}$.
64. $(\frac{4}{9})^{\frac{1}{2}}$.
65. $(-\frac{8}{27})^{\frac{1}{3}}$.
66. $(\frac{8}{27})^{\frac{2}{3}}$.
67. $(\frac{1}{4})^{\frac{3}{2}}$.
68. $(.125)^{\frac{2}{3}}$.
69. $(.027)^{\frac{2}{3}}$.

Illustrative example. Find the value of $8^{\frac{2}{3}} - 32^{\frac{2}{5}} + 1^{\frac{3}{4}} - 16^{\frac{3}{4}}$.

Solution

$$\begin{aligned} 8^{\frac{2}{3}} - 32^{\frac{2}{5}} + 1^{\frac{3}{4}} - 16^{\frac{3}{4}} &= \sqrt[3]{8^2} - \sqrt[5]{32^2} + \sqrt[4]{1^3} - \sqrt[4]{16^3} \\ &= 2^2 - 2^2 + 1^3 - 2^3 \\ &= -7. \end{aligned}$$

70. $9^{\frac{3}{2}} + 64^{\frac{1}{3}} - 1^{\frac{2}{3}}$.
71. $27^{\frac{1}{3}} - 4^{\frac{3}{2}} + (-8)^{\frac{2}{3}}$.
72. $25^{\frac{3}{2}} + 1^{\frac{1}{2}} - (-27)^{\frac{2}{3}}$.
73. $(\frac{4}{9})^{\frac{1}{2}} + (\frac{8}{27})^{\frac{2}{3}} + (\frac{16}{81})^{\frac{3}{4}}$.
74. $.027^{\frac{1}{3}} - .125^{\frac{2}{3}} + .09^{\frac{1}{2}}$.
75. $(.027)^{\frac{2}{3}} + (.064)^{\frac{2}{3}} - .125^{\frac{1}{3}}$.

The meaning of a zero exponent. According to the law,

$$x^m \div x^n = x^{m-n},$$

$$\frac{a^4}{a^4} = a^{4-4} = a^0; \text{ but } \frac{a^4}{a^4} = 1.$$

$\therefore$ we define a^0 as equal to 1.

In general $x^m \div x^m = x^{m-m} = x^0$; but $\dfrac{x^m}{x^m} = 1$.

$$\therefore \mathbf{x^0 = 1.}$$

In other words: *Any quantity (except zero) with a zero exponent is equal to 1.*

Thus: $5^0 = 1$; $m^0 = 1$; $(-\frac{1}{3})^0 = 1$; $(2a + b)^0 = 1$.

EXERCISES

Find the value of each of the following:

1. 100^0.
2. $(\frac{2}{3})^0$.
3. $(-2)^0$.
4. -2^0.
5. $3a^0$.
6. $(-3a)^0$.
7. $-3a^0$.
8. $-(3a)^0$.
9. $(a+b)^0$.
10. $a^0 - b$.
11. $\dfrac{1}{x + y^0}$.
12. $\dfrac{1}{x^0 + y^0}$.
13. $\dfrac{y^0 + 2}{3 + x^0}$.

The meaning of a negative exponent. According to the law,

$$x^m \div x^n = x^{m-n},$$

$$x^5 \div x^7 = x^{5-7} = x^{-2}; \text{ but } \frac{x^5}{x^7} = \frac{1}{x^2}.$$

$$\therefore x^{-2} = \frac{1}{x^2}.$$

Again $\quad x^3 \div x^4 = x^{3-4} = x^{-1}$; but $\dfrac{x^3}{x^4} = \dfrac{1}{x}$.

$$\therefore x^{-1} = \frac{1}{x}.$$

In general $x^n \div x^{2n} = x^{n-2n} = x^{-n}$; but $\dfrac{x^n}{x^{2n}} = \dfrac{1}{x^n}$.

$$\therefore \mathbf{x^{-n} = \frac{1}{x^n}.}$$

In other words: *Any quantity with a negative exponent is equal to 1 divided by the same quantity with a positive exponent having the same absolute value.*

Thus, $x^{-5} = \dfrac{1}{x^5}$; $y^{-\frac{1}{2}} = \dfrac{1}{y^{\frac{1}{2}}}$; $(a+b)^{-2} = \dfrac{1}{(a+b)^2}$;

$$(a-b)^{-\frac{1}{2}} = \frac{1}{(a-b)^{\frac{1}{2}}}.$$

$(\sin M)^{-1}$ must not be confused with $\sin^{-1} M$. As indicated above, $(\sin M)^{-1} = \dfrac{1}{\sin M}$ and represents a fraction, whereas $\sin^{-1} M$ means "the angle whose sine is M" and represents measures of angles in degrees or radians.

EXERCISES

Write with positive exponents and simplify if possible:

1. m^{-2}.	**5.** $(ab)^{-3}$.	**9.** 4^{-2}.	**13.** $(\frac{1}{2})^{-2}$.
2. x^{-5}.	**6.** x^{-a}.	**10.** 2^{-4}.	**14.** $(\frac{2}{3})^{-2}$.
3. a^{-7}.	**7.** a^{-x}.	**11.** 2^{-2}.	**15.** $(\frac{1}{4})^{-3}$.
4. b^{-1}.	**8.** 2^{-1}.	**12.** 5^{-2}.	**16.** $(\frac{2}{5})^{-3}$.

Using negative exponents. Negative exponents may be used to simplify exponential expressions.

Illustrative examples.

Example 1. Simplify $\dfrac{x^{-m}}{y^{-n}}$.

Solution

Changing from negative to positive exponents:

$$\frac{x^{-m}}{y^{-n}} = \frac{\frac{1}{x^m}}{\frac{1}{y^n}} = \frac{1}{x^m} \div \frac{1}{y^n} = \frac{1}{x^m} \cdot \frac{y^n}{1} = \frac{y^n}{x^m}.$$

Note that moving x^{-m} down to the denominator and y^{-n} up to the numerator caused the sign of each exponent to be changed from minus to plus.

Example 2. Simplify $\dfrac{m^3x^{-2}}{a^{-3}b}$.

Solution

$$\frac{m^3x^{-2}}{a^{-3}b} = \frac{m^3 \cdot \dfrac{1}{x^2}}{\dfrac{1}{a^3} \cdot b} = \frac{\dfrac{m^3}{x^2}}{\dfrac{b}{a^3}} = \frac{m^3}{x^2} \times \frac{a^3}{b} = \frac{m^3a^3}{x^2b}.$$

Which factor was moved down to the denominator? Which factor was moved up to the numerator? What effect had this upon the signs of their exponents?

Example 3. Write $\dfrac{a^2b^3}{c^4}$ without a denominator.

Solution

$$\frac{a^2b^3}{c^4} = \frac{a^2b^3}{1} \times \frac{1}{c^4}. \qquad (1)$$

But $$\frac{1}{c^4} = c^{-4}.$$

Substituting in (1) $$\frac{a^2b^3}{c^4} = a^2b^3c^{-4}.$$

Any factor may be transferred from the numerator to the denominator or from the denominator to the numerator provided the sign of its exponent is changed.

Example 4. Simplify $\dfrac{7\,a^5x^{-\frac{5}{3}}}{3\,a^{-5}x^{-\frac{4}{3}}}$.

Solution

$$\frac{7\,a^5x^{-\frac{5}{3}}}{3\,a^{-5}x^{-\frac{4}{3}}} = \frac{7\,a^5 \times a^5 \times \cancel{x^{\frac{4}{3}}}}{3\,\underset{x^{\frac{1}{3}}}{\cancel{x^{\frac{5}{3}}}}} = \frac{7\,a^{10}}{3\,x^{\frac{1}{3}}}.$$

Example 5. Write $\dfrac{2}{a^{-2}+b^{-2}}$ with positive exponents and simplify.

Solution

Observe in this case that neither a^{-2} nor b^{-2} is a factor of the denominator and therefore cannot be transferred directly to the numerator as in illustrative example 4.

Thus $$\frac{2}{a^{-2}+b^{-2}}=\frac{2}{\frac{1}{a^2}+\frac{1}{b^2}}=\frac{2}{\frac{b^2+a^2}{a^2b^2}}=\frac{2\,a^2b^2}{a^2+b^2}.$$

EXERCISES

Express with positive exponents and simplify:

1. $\dfrac{a^{-3}}{b^{-2}}$.
2. $\dfrac{x^3}{y^{-2}}$.
3. $\dfrac{m^{-3}}{x^2}$.
4. $3\,a^{-2}$.
5. $(3\,a)^{-2}$.
6. $2(3\,a)^{-2}$.
7. $\dfrac{2\,a^{-3}}{b^{-2}}$.
8. $\dfrac{2\,a}{b^{-3}}$.
9. $3\,a^{-3}x^4$.
10. $7\,a^{-4}bc^{-2}$.
11. $\dfrac{-3\,a^{-3}}{b^2}$.
12. $\dfrac{3^{-1}a^3}{b^{-2}}$.
13. $\dfrac{3\,x^{-3}y}{2^{-1}\,x^2}$.
14. $\dfrac{x^2y^{-5}}{2^{-2}\,x^{-1}y}$.
15. $\dfrac{9^{\frac{1}{2}}\,x^{-3}}{ay^{-5}}$.

Write without denominators:

16. $\dfrac{ab}{c}$.
17. $\dfrac{2\,x^2}{y}$.
18. $\dfrac{a^{-1}}{xy^2}$.
19. $\dfrac{2}{a^{-2}b}$.
20. $\dfrac{1}{a^4b^2}$.
21. $\dfrac{1}{2\,a^3b^2}$.
22. $\dfrac{m^4}{3\,a^{-1}b^5}$.
23. $\dfrac{x^{\frac{1}{2}}m^{-2}}{2\,a^{-1}b}$.
24. $\dfrac{2\,a^2b^{-1}}{3\,m^{-2}n}$.
25. $\dfrac{a^2b^{-3}}{2^{-2}m^{-1}n^2}$.
26. $\dfrac{5\,x^{-1}y^{-2}}{3\,ab}$.
27. $\dfrac{a^6b^3c^{-2}}{m^2n^{-2}y^3}$.

28. $\dfrac{a^{-\frac{2}{3}}x^{\frac{1}{6}}}{b^{-\frac{1}{2}}c^{\frac{3}{8}}}$. 29. $\dfrac{y^{-3}}{27^{\frac{1}{3}}\,x^4}$. 30. $\dfrac{27^{-\frac{1}{3}}\,x^4y^{-\frac{1}{2}}}{abc}$.

Write with radical signs and positive exponents:

31. $m^{\frac{1}{2}}$.
32. $m^{-\frac{1}{2}}$.
33. $2\,m^{\frac{1}{2}}$.
34. $2\,m^{-\frac{1}{2}}$.
35. $(2\,m)^{\frac{1}{2}}$.
36. $(2m)^{-\frac{1}{2}}$.
37. $\dfrac{1}{m^{\frac{1}{2}}}$.
38. $\dfrac{1}{m^{-\frac{1}{2}}}$.
39. $\dfrac{1}{2\,m^{\frac{1}{2}}}$.
40. $\dfrac{1}{2\,m^{-\frac{1}{2}}}$.
41. $\dfrac{1}{(2\,m)^{\frac{1}{2}}}$.
42. $\dfrac{1}{(2\,m)^{-\frac{1}{2}}}$.
43. $\dfrac{2}{m^{\frac{1}{2}}}$.
44. $\dfrac{2}{m^{-\frac{1}{2}}}$.
45. $a^{\frac{2}{3}}$.
46. $2^{-\frac{1}{2}}$.
47. $3^{-\frac{1}{3}}$.
48. $4^{-\frac{2}{3}}$.
49. $\dfrac{1}{2^{-\frac{1}{2}}}$.
50. $\dfrac{3}{3^{-\frac{1}{2}}}$.

Find the value of each of the following:

51. $64^{-\frac{2}{3}} - 3(13)^0 + 12(2)^{-2}$.
52. $27^{-\frac{1}{3}} - 3\,a^0 - 1^{\frac{3}{2}} + \dfrac{1}{8^{-\frac{2}{3}}}$.
53. $8^{-\frac{2}{3}} \times 16^{\frac{3}{4}} \times 2^0$.
54. $3^0 \times 4^{\frac{1}{2}} \times 16^{-\frac{3}{4}}$.
55. $\dfrac{3}{2 + 2^{-1}}$.
56. $\dfrac{3^{-2} + 5(2)^0}{3 - 4(3)^{-1}}$.
57. $\dfrac{1}{4^{-1} + 2^{-2}}$.
58. $\dfrac{3^0\,x + 4\,x^{-1}}{x^{\frac{2}{3}}}$ when $x = 8$.
59. $\dfrac{x^{\frac{2}{3}} - 3\,x^0}{x^{-\frac{1}{6}}}$ when $x = 64$.
60. $\dfrac{(5\,x)^0 + x^{-\frac{1}{3}}}{\sqrt[3]{-x}}$ when $x = 27$

Multiplication with exponents. You know that:

$(-2)^4 = +16$. $(+2)^4 = +16$.
$(-2)^3 = -8$. $(+2)^3 = +8$.

Recall fact 57. *A negative quantity raised to an even power is positive.*

Recall fact 58. *A negative quantity raised to an odd power is negative.*

Recall fact 59. *A positive quantity raised to an even or an odd power is positive.*

Illustrative examples.

Example 1. $[x^m \times x^n = x^{m+n}.]$

(a) $a^4 \cdot a^{-2} = a^{4-2} = a^2.$

(b) $2\,b^{-6} \cdot 3\,b^2 = 6\,b^{-4}.$

(c) $3\,b^{-3} \cdot 2\,b^3 = 6 \times b^0 = 6 \times 1 = 6.$

(d) $xy^2 \cdot x^3y^{-2} = x^4y^0 = x^4.$

(e) $x^{\frac{1}{2}} \cdot x^{\frac{3}{2}} = x^{\frac{1}{2}+\frac{3}{2}} = x^2.$

Example 2. $[(x^m)^n = x^{mn}.]$

(a) $(a^2)^3 = a^6.$

(b) $(a^2)^{\frac{1}{2}} = a.$

(c) $(x^{\frac{3}{4}})^{\frac{4}{3}} = x.$

(d) $(a^4)^{-2} = a^{-8}.$

Example 3. $[(x^ay^b)^m = x^{am}y^{bm}.]$

(a) $(a^2b^2)^3 = a^6b^6.$

(b) $(x^{-2}y)^2 = x^{-4}y^2.$

(c) $(xy^{-3})^{-2} = x^{-2}y^6.$

(d) $(a^{-\frac{1}{2}}y^{\frac{1}{3}})^{\frac{2}{3}} = a^{-\frac{1}{3}}y^{\frac{2}{9}}.$

(e) $(x^{-2}y^{\frac{3}{2}})^{-2} = x^4y^{-3}.$

Example 4. Multiply $3\,a^{\frac{1}{2}} - 2\,a^{\frac{1}{4}} + 3$ by $3 + a^{-\frac{1}{4}} - 2\,a^{-\frac{1}{2}}.$

Solution

$$\begin{array}{l}
3\,a^{\frac{1}{2}} - 2\,a^{\frac{1}{4}} + 3 \\
\quad 3 + a^{-\frac{1}{4}} - 2\,a^{-\frac{1}{2}} \\
\hline
9\,a^{\frac{1}{2}} - 6\,a^{\frac{1}{4}} + 9 \\
\qquad + 3\,a^{\frac{1}{4}} - 2 + 3\,a^{-\frac{1}{4}} \\
\qquad\qquad\qquad - 6 + 4\,a^{-\frac{1}{4}} - 6\,a^{-\frac{1}{2}} \\
\hline
9\,a^{\frac{1}{2}} - 3\,a^{\frac{1}{4}} + 1 + 7\,a^{-\frac{1}{4}} - 6\,a^{-\frac{1}{2}}.
\end{array}$$

EXERCISES

Perform the indicated multiplications:

1. $a^3 \times a^{-4}$.
2. $x^{-2} \times x^{-5}$.
3. $2\,a^2 \times 3\,a^{-4}$.
4. $a^{-4} \times 2\,a$.
5. $3\,a^{-3} \times 2\,a$.
6. $x^4 \times x^{-3}$.
7. $m^2 \times m^{-2}$.
8. $3\,x^2 \times x^{-2}$.
9. $2\,y^3 \times 3\,y^{-3}$.
10. $x^{\frac{1}{2}} \times x^{-\frac{3}{2}}$.
11. $x^{-\frac{1}{2}} \times x^{-\frac{1}{2}}$.
12. $x^{\frac{5}{2}} \times x^{-\frac{3}{2}}$.
13. $a^{\frac{1}{2}} \times a^{-\frac{1}{2}}$.
14. $2\,m^{-\frac{1}{2}} \times m^{\frac{1}{2}}$.
15. $5\,x^{\frac{2}{3}} \times 2\,x^{-\frac{2}{3}}$.
16. $3\,a^{\frac{3}{4}} \times 5\,a^{\frac{3}{4}}$.

Simplify:

17. $(a^3)^4$.
18. $(b^2)^{-5}$.
19. $(b^5)^2$.
20. $(b^2)^{-5}$.
21. $(b^{-2})^5$.
22. $(b^{-2})^{-5}$.
23. $(-\,a^3)^4$.
24. $(-\,a^{-3})^4$.
25. $(a^3)^{\frac{1}{2}}$.
26. $(a^{-3})^{\frac{1}{2}}$.
27. $(m^{-3})^{\frac{1}{2}}$.
28. $(a^{\frac{1}{2}})^3$.
29. $(b^{-\frac{1}{2}})^3$.
30. $(c^{\frac{1}{2}})^{\frac{2}{3}}$.
31. $(c^{-\frac{1}{2}})^{-\frac{2}{3}}$.
32. $(d^{-\frac{2}{3}})^{\frac{1}{3}}$.
33. $(a^2b^3)^4$.
34. $(a^2b^3)^{\frac{1}{2}}$.
35. $(a^{-1}b^2)^2$.
36. $(x^{-2}y^{-3})^3$.
37. $(p^2x^5)^{-2}$.
38. $(p^{-2}x^{-3})^{-2}$.
39. $(a^{\frac{1}{2}}b^{\frac{1}{2}})^2$.
40. $(-\,a^{\frac{3}{4}}c^{\frac{1}{4}})^2$.
41. $(-\,a^{-\frac{1}{4}}b^{\frac{1}{3}})^3$.
42. $(x^{\frac{1}{2}}y^{\frac{2}{3}})^{\frac{1}{2}}$.
43. $(m^{\frac{1}{4}}y^{\frac{1}{2}})^{-\frac{3}{4}}$.
44. $\left(\dfrac{4\,a^{-1}}{9}\right)^{-2}$.
45. $(x^{-\frac{1}{2}}y^{-1})^{-2}$.
46. $(m^{\frac{1}{2}}b^{-6})^{-\frac{1}{2}}$.
47. $(a^{-\frac{1}{2}}b^{\frac{1}{5}})^{-\frac{2}{5}}$.
48. $(-\,2\,a^3)^2$.
49. $(4\,m^{-1})^{-3}$.
50. $(9\,x^{\frac{1}{2}})^{\frac{1}{2}}$.
51. $(9\,x^{-\frac{1}{2}})^{-\frac{1}{2}}$.
52. $\left(\dfrac{2\,a}{3}\right)^2$.
53. $\left(\dfrac{27\,a^{\frac{1}{2}}}{b^{-\frac{1}{4}}}\right)^{-\frac{1}{3}}$.

Perform the indicated multiplications:

54. $(2\,x + 4\,x^{-1} - 3)(3\,x^{-1} + 4 - 2\,x)$.
55. $(3\,x^{-2} + 2\,x^{-1})(2\,x^{-1} + 3 - x)$.
56. $(x^{-m} + x^m + 1)(x^{-m} + x^m - 1)$.
57. $(2\,x^2 - x + 4 - 3\,x^{-1})^2$.
58. $(m^{-2} + 3\,m^{-1} + 2 - m)^2$.

59. $(a^{\frac{1}{3}} + 1 + a^{-\frac{1}{3}})(a^{\frac{2}{3}} + a^{-\frac{2}{3}})$.

60. $(m^{\frac{2}{3}} - m^{\frac{1}{3}}n^{\frac{1}{3}} + n^{\frac{2}{3}})(m^{\frac{1}{3}} + n^{\frac{1}{3}})$.

61. $(x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}})$.

62. $(2\,a^{\frac{1}{2}} - 3\,b^{\frac{1}{2}})^2$.

63. $(2\,a^{\frac{1}{2}}b^{\frac{1}{3}} - 3\,a^{-\frac{1}{2}}b^{\frac{2}{3}})^2$.

64. $(e^x + e^{-x})^2$.

65. $(x + x^{\frac{1}{2}} + 1 + x^{-\frac{1}{2}})^2$.

Division with exponents. Recall the law $x^m \div x^n = x^{m-n}$.

Illustrative example.

(*a*) $x^5 \div x^6 = x^{5-6} = x^{-1}$.

(*b*) $x^{-1} \div x^5 = x^{-1-5} = x^{-6}$.

(*c*) $x^{-2} \div x^{-2} = x^{-2+2} = x^0 = 1$.

(*d*) $x^{\frac{5}{2}} \div x = x^{\frac{5}{2}-\frac{2}{2}} = x^{\frac{3}{2}}$.

EXERCISES

Perform the indicated divisions:

1. $x^4 \div x^5$.
2. $x^4 \div x^{-2}$.
3. $a^2 \div a^{\frac{1}{2}}$.
4. $m^2 \div m^{-\frac{1}{2}}$.
5. $m^{-2} \div m^{-\frac{1}{2}}$.
6. $x^{\frac{1}{2}} \div x^{\frac{1}{2}}$.
7. $x^{\frac{1}{2}} \div x^{-1}$.
8. $y^{\frac{1}{3}} \div y^{-\frac{2}{3}}$.
9. $b^{\frac{1}{2}} \div b^{\frac{1}{3}}$.
10. $6\,a^{\frac{2}{3}} \div 2\,a^{\frac{1}{3}}$.
11. $8\,x \div 4\,x^{\frac{1}{3}}$.
12. $10\,a \div 5\,a^{\frac{1}{2}}$.
13. $6\,a^{\frac{3}{2}} \div 2\,a^{-\frac{1}{2}}$.
14. $x^{2a} \div x^a$.
15. $x^{a+2} \div x^{a-2}$.

Arrangement of terms in division of polynomials. In dividing polynomials, you have learned to arrange the terms of the dividend and divisor in either ascending or descending powers of the same letter. This is equally true when fractional or negative exponents are involved. Below are powers of x arranged in descending order:

$$x^4,\quad x^3,\quad x^2,\quad x,\quad 1(x^0),\quad x^{-1},\quad x^{-2},\quad x^{-3}$$

$$x^2,\quad x^{\frac{3}{2}},\quad x,\quad x^{\frac{1}{2}},\quad x^0,\quad x^{-\frac{1}{2}},\quad x^{-\frac{3}{2}},\quad x^{-2}$$

Since $x^0 = 1$ and $4\,x^0 = 4$, a constant term may be regarded as the coefficient of the unknown letter with a zero exponent. Observe also that terms having negative exponents are lower powers of the unknown than those having a zero exponent.

EXERCISES

Arrange the following in (*a*) descending and (*b*) ascending powers of x.

1. $x^3 + x^{-3} + x^2 - x^{-2} + x - x^{-1} + 5.$
2. $x^3 - xy - y^{\frac{3}{2}} + x^2y^{\frac{1}{2}}.$
3. $x^2 + a^2x^{-2} - 2\,a^{\frac{1}{2}}x - 2\,a^{\frac{3}{2}}x^{-1} + 3\,a.$
4. $9\,x^2 - 20\,x^{\frac{1}{2}} + 25 + 34\,x - 12\,x^{\frac{3}{2}}.$
5. $6\,x + 6 + x^{\frac{1}{2}} - 3\,x^{\frac{3}{4}} - 4\,x^{\frac{1}{4}} - x^{-\frac{1}{4}} + 3\,x^{-\frac{1}{2}}.$

Division of polynomials.

Illustrative examples.

Example 1. Divide $\dfrac{3}{x^4} - x^4 - 8 + \dfrac{13}{x^2} - 7\,x^2$ by $\dfrac{3}{x^2} - 2 - x^2.$

Solution

$$\begin{array}{rrrrr|l}
3\,x^{-4} & +\,13\,x^{-2} & -\;\;8 & -\,7\,x^2 & -\,x^4 & 3\,x^{-2} - 2 - x^2 \\ \cline{6-6}
3\,x^{-4} & -\;\;2\,x^{-2} & -\;\;1 & & & x^{-2} + 5 + x^2 \,* \quad \text{Ans.} \\ \cline{1-3}
 & 15\,x^{-2} & -\;\;7 & -\,7\,x^2 & & \\
 & 15\,x^{-2} & -\,10 & -\,5\,x^2 & & \\ \cline{2-5}
 & & 3 & -\,2\,x^2 & -\,x^4 & \\
 & & 3 & -\,2\,x^2 & -\,x^4 & \\ \cline{3-5}
\end{array}$$

$$*\left[\frac{3}{3\,x^{-2}} = \frac{1}{x^{-2}} = x^2\right]$$

Example 2. Divide $\sqrt[5]{a^2} + \dfrac{2\sqrt[5]{a}}{\sqrt{b}} + \dfrac{1}{b}$ by $\sqrt[5]{a} + \dfrac{1}{\sqrt{b}}.$

Solution

$$\begin{array}{rrr|l}
a^{\frac{2}{5}} & +\,2\,a^{\frac{1}{5}}b^{-\frac{1}{2}} & +\,b^{-1} & a^{\frac{1}{5}} + b^{-\frac{1}{2}} \\ \cline{4-4}
a^{\frac{2}{5}} & +\;\;a^{\frac{1}{5}}b^{-\frac{1}{2}} & & a^{\frac{1}{5}} + b^{-\frac{1}{2}} \quad \text{Ans.} \\ \cline{1-3}
 & +\;\;a^{\frac{1}{5}}b^{-\frac{1}{2}} & +\,b^{-1} & \\
 & +\;\;a^{\frac{1}{5}}b^{-\frac{1}{2}} & +\,b^{-1} & \\ \cline{1-3}
\end{array}$$

EXERCISES

Perform the indicated divisions:

1. $(3x + 7 - x^{-1} - x^{-2}) \div (3 + x^{-1})$.
2. $(6x^{-1} + 3x^{-3} + x^{-2} + 2x^{-4}) \div (2x^{-1} + x^{-2})$.
3. $(2a^2 - a^{-2} - 13 + 7a^{-1} + 2a) \div (a + 3 - a^{-1})$.
4. $(1 + 6a - a^{-3} + 3a^{-2} - a^{-1}) \div (2a + a^{-1} - 1)$.
5. $\left(\frac{12}{a^2} + \frac{11}{a} - \frac{1}{a^3} - 21 - \frac{1}{a^4}\right) \div \left(7 + \frac{1}{a} - \frac{1}{a^2}\right)$.
6. $(22x - 12x^2 - 30 + x^{-1} + 15x^{-2}) \div (2x^2 - 3x + 5)$.
7. $(a^4b^{-4} - a^{-4}b^4 - 4 - 4a^{-2}b^2) \div (a^2b^{-2} + a^{-2}b^2 + 2)$.
8. $\left(e^{2x} - 2 + \frac{1}{e^{2x}}\right) \div \left(e^x - \frac{1}{e^x}\right)$.
9. $(x^{6a} - y^{6b}) \div (x^{2a} - y^{2b})$.
10. $(a^{-3} - b^{-3}) \div (a^{-1} - b^{-1})$.
11. $(a^{\frac{3}{4}} - 3a^{\frac{1}{4}}b^{\frac{1}{2}} + a^{\frac{1}{2}}b^{\frac{1}{4}} + b^{\frac{3}{4}}) \div (a^{\frac{1}{4}} - b^{\frac{1}{4}})$.
12. $(9x^2 - 12\sqrt{x^3} + 34x - 20\sqrt{x} + 25) \div (3x - 2\sqrt{x} + 5)$.

Write the quotients of the following without negative and fractional exponents.

13. $(a^2 + b^2) \div (a^{\frac{2}{3}} + b^{\frac{2}{3}})$.   14. $(x^2 - y^2) \div (x^{\frac{1}{3}} - y^{\frac{1}{3}})$.
15. $(a^{-\frac{4}{3}} - a^{-1}b^{-1} - 3a^{-\frac{2}{3}}b^{-2} + 2a^{-\frac{1}{3}}b^{-3}) \div (a^{-\frac{1}{3}} - 2b^{-1})$.

Simplification of more difficult expressions involving negative and fractional exponents. We shall be able to save considerable time in simplifying expressions involving exponents by observing the following:

Step 1. Transform from radical to fractional exponent form.
Step 2. Perform the operations indicated.
Step 3. Change negative exponents to positive exponents.
Step 4. Simplify, if possible.

Illustrative examples.

Example 1. Simplify $\sqrt[3]{\left(\dfrac{a^2x}{64\,a^{-3}x^{\frac{1}{3}}}\right)^{-1}}$.

Solution

Step 1. $\sqrt[3]{\left(\dfrac{a^2x}{64\,a^{-3}x^{\frac{1}{3}}}\right)^{-1}} = \left(\dfrac{a^2x}{64\,a^{-3}x^{\frac{1}{3}}}\right)^{-\frac{1}{3}}$

Step 2. $= \dfrac{a^{-\frac{2}{3}}x^{-\frac{1}{3}}}{64^{-\frac{1}{3}}\,ax^{-\frac{1}{9}}}$

Step 3. $= \dfrac{64^{\frac{1}{3}}\,x^{\frac{1}{9}}}{aa^{\frac{2}{3}}x^{\frac{1}{3}}}.$

Step 4. $= \dfrac{4}{a^{\frac{5}{3}}x^{\frac{2}{9}}}.$

Example 2. Simplify $\sqrt[5]{\left(\dfrac{x^{-2}y^3}{x^3y^{-2}}\right)^{-1}} \cdot \left(\dfrac{y^3x^{-3}}{x^3y^{-3}}\right)^{-1}$.

Solution

$$\sqrt[5]{\left(\frac{x^{-2}y^3}{x^3y^{-2}}\right)^{-1}} \cdot \left(\frac{y^3x^{-3}}{x^3y^{-3}}\right)^{-1} = \left(\frac{x^{-2}y^3}{x^3y^{-2}}\right)^{-\frac{1}{5}} \cdot \left(\frac{y^3x^{-3}}{x^3y^{-3}}\right)^{-1}$$
$$= \frac{x^{\frac{2}{5}}y^{-\frac{3}{5}}}{x^{-\frac{3}{5}}y^{\frac{2}{5}}} \cdot \frac{y^{-3}x^3}{x^{-3}y^3}$$
$$= \frac{x^{\frac{2}{5}}x^{\frac{3}{5}}}{y^{\frac{3}{5}}y^{\frac{2}{5}}} \cdot \frac{x^3x^3}{y^3y^3}$$
$$= \frac{x}{y} \cdot \frac{x^6}{y^6} = \frac{x^7}{y^7}.$$

EXERCISES

Simplify each of the following:

1. $(\sqrt[5]{x})^5$.
2. $(\sqrt[3]{m})^{\frac{3}{2}}$.
3. $\sqrt[4]{\sqrt[3]{x}}$.
4. $\sqrt{(25\,a^{\frac{2}{3}}b^{-4})^3}$.
5. $\sqrt[3]{\dfrac{27\,x^3}{a^3b^6}}$.
6. $\sqrt{\dfrac{x^{-2}y^6}{9}}$.
7. $\sqrt[3]{(-8\,x^3y^{-\frac{3}{2}})^2}$.
8. $\sqrt[4]{\dfrac{x^4y^2}{16\,m^{-8}n^4}}$.

9. $\sqrt[3]{\left(\dfrac{64\,m^6}{27\,n^{-3}}\right)^{-1}}$. 11. $\dfrac{x^{\frac{1}{3}}\sqrt{y^{-\frac{3}{2}}}}{y^{\frac{3}{4}}\sqrt[3]{x^{-1}}}$. 13. $\dfrac{a^{-1}b\sqrt{3}}{a^{\frac{2}{3}}} \div \sqrt{\dfrac{a^2b^{-1}}{c^{-3}}}$.

10. $\sqrt{\left(\dfrac{ab^{-3}}{a^{-3}b^2c}\right)^{-1}}$. 12. $\left(\sqrt[5]{x^{\frac{4}{3}}}\right)^{-\frac{5}{3}}$. 14. $\sqrt[n]{\dfrac{32}{2^{5+n}}}$.

Multiplication and division of radicals of different root indices. So far we have learned how to multiply and divide radicals having the same root index. Thus, $\sqrt{2} \times \sqrt{3} = \sqrt{6}$. Now we shall learn how to multiply $\sqrt[3]{2}$ by $\sqrt{3}$, *i.e.*, radicals with different root indices. We have learned how to incorporate radicands with the same index under one radical sign, but not how to incorporate radicands with different root indices under one radical sign.

Illustrative examples.

Example 1. Multiply $\sqrt[3]{2}$ by $\sqrt{3}$.

Analysis

$\sqrt[3]{2} = 2^{\frac{1}{3}}$ and $\sqrt{3} = 3^{\frac{1}{2}}$. Since the denominators of the fractions are the root indices, we must change these fractions to equivalent fractions having the same denominator, in this case, the denominator 6.

Solution

$$\therefore\ \sqrt[3]{2} = 2^{\frac{1}{3}} = 2^{\frac{2}{6}} = \sqrt[6]{2^2} = \sqrt[6]{4},$$

and

$$\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{3}{6}} = \sqrt[6]{3^3} = \sqrt[6]{27}.$$

$$\therefore\ \sqrt[3]{2} \cdot \sqrt{3} = \sqrt[6]{4} \cdot \sqrt[6]{27} = \sqrt[6]{108}.$$

Example 2. Divide $\sqrt[4]{3}$ by $\sqrt{2}$.

Solution

$$\frac{\sqrt[4]{3}}{\sqrt{2}} = \frac{3^{\frac{1}{4}}}{2^{\frac{1}{2}}} = \frac{3^{\frac{1}{4}}}{2^{\frac{2}{4}}} = \frac{\sqrt[4]{3}}{\sqrt[4]{2^2}} = \frac{\sqrt[4]{3}}{\sqrt[4]{4}} = \sqrt[4]{\frac{3}{4}}$$

$$= \sqrt[4]{\tfrac{3}{4} \cdot \tfrac{4}{4}} = \sqrt[4]{\tfrac{12}{16}} = \sqrt[4]{\tfrac{1}{16} \cdot 12} = \tfrac{1}{2}\sqrt[4]{12}.$$

EXERCISES

Perform the operations indicated:

1. $\sqrt[6]{2} \cdot \sqrt[6]{3}$.
2. $\sqrt[5]{5} \cdot \sqrt[5]{2}$.
3. $2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}$.
4. $4^{\frac{1}{3}} \cdot 3^{\frac{1}{3}}$.
5. $\sqrt[3]{4} \cdot \sqrt{3}$.
6. $\sqrt{2} \cdot \sqrt[3]{3}$.
7. $\sqrt[3]{4} \cdot \sqrt{2}$.
8. $\sqrt[3]{6} \cdot \sqrt{\frac{2}{3}}$.
9. $\sqrt[4]{10} \cdot \sqrt[3]{2}$.
10. $\sqrt[3]{3} \cdot \sqrt{8}$.
11. $\sqrt[3]{10} \div \sqrt[3]{2}$.
12. $\sqrt[6]{12} \div \sqrt[6]{2}$.
13. $2^{\frac{3}{4}} \div 2^{\frac{1}{4}}$.
14. $\sqrt{3} \div \sqrt[3]{4}$.
15. $\sqrt[4]{2} \div \sqrt{5}$.

Exponential Equations

Solving exponential equations. Frequently it is necessary to solve an equation containing fractional or negative exponents or an equation in which the unknown quantity is an exponent. The latter type of equation is called an *exponential equation.* The law of exponents, $(x^m)^n = x^{mn}$, will help us to solve exponential equations.

Illustrative examples.

Example 1. Solve $x^{\frac{2}{3}} = 4$.

Solution

Since it is required to find the value of x having the exponent 1, and since $\frac{2}{3} \times \frac{3}{2} = 1$, we must raise both members of the equation to the $\frac{3}{2}$ power.

Thus, $(x^{\frac{2}{3}})^{\frac{3}{2}} = (4)^{\frac{3}{2}}$.

$\therefore x = 4^{\frac{3}{2}} = 8$.

Check

$8^{\frac{2}{3}} = 4$.

$4 = 4$.

Example 2. Solve $x^{-\frac{3}{4}} = 8$.

Solution

Raising both members to the $-\frac{4}{3}$ power, we have

$x = (x^{-\frac{3}{4}})^{-\frac{4}{3}} = (8)^{-\frac{4}{3}} = \dfrac{1}{16}$.

Check $(\frac{1}{16})^{-\frac{3}{4}} = 8$.

$8 = 8$.

Example 3. Solve $2^x = 32$.

Solution

$$\begin{aligned} \text{Since} \quad 32 &= 2^5,\ 2^x = 2^5. \\ \text{As} \quad 2 &= 2, \text{ the exponents must be equal.} \\ \therefore\ x &= 5. \end{aligned}$$

Example 4. Solve $3^{x-1} = 81$.

Solution

$$\begin{aligned} 3^{x-1} &= 3^4, \text{ since } 81 = 3^4. \\ \therefore\ x - 1 &= 4. \\ x &= 5. \end{aligned}$$

EXERCISES

Solve each of the following equations and check:

1. $x^{\frac{1}{2}} = 4$.

2. $x^{\frac{1}{4}} = 2$.

3. $x^{\frac{1}{3}} = \frac{1}{2}$.

4. $x^{\frac{1}{3}} = 2$.

5. $x^{\frac{3}{4}} = 8$.

6. $x^{\frac{3}{2}} = 27$.

7. $x^{\frac{5}{2}} = 32$.

8. $x^{\frac{2}{3}} - 1 = 15$.

9. $x^{\frac{1}{2}} = m$.

10. $x\sqrt{x} = b$.

11. $x^{-\frac{1}{2}} = 2$.

12. $x^{-\frac{1}{4}} = 2$.

13. $x^{-\frac{1}{3}} = \frac{1}{3}$.

14. $x^{-\frac{2}{3}} = 4$.

15. $x^{-\frac{2}{5}} = 9$.

16. $x^{-\frac{3}{4}} = 64$.

17. $x^{-\frac{1}{3}} - 2 = 1$.

18. $x^{-\frac{3}{2}} + 1 = 9$.

19. $x^{-\frac{2}{3}} = a$.

20. $x^{-\frac{2}{5}} = n$.

21. $x^{-\frac{3}{4}} = k$.

22. If $x^2 = 64$, find the value of x^{-3}.

23. If $y^{\frac{3}{4}} = 8$, find the value of $y^{\frac{1}{2}}$.

24. If $2\,x^{-\frac{1}{2}} = 1$, find the value of $x^{\frac{5}{2}}$.

Solve each of the following exponential equations and check:

25. $2^x = 2^4$.

26. $2^{x+1} = 128$.

27. $2^{x-1} = 2^{1-x}$.

28. $3^x = 81$.

29. $3^{1-x} = 81$.

30. $2^x = \frac{1}{4}$.

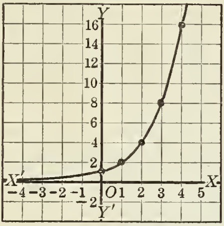
Graphing exponential equations. Exponential equations may be graphed in exactly the same way as any other equation.

Illustrative example. Draw the graph of $y = 2^x$, from $x = -4$ to $x = +4$.

Solution

If $x =$	Then $y =$
-4	$\frac{1}{16}$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8
4	16

Drawing the graph, we have



EXERCISES

Draw the graph of:

1. $y = 3^x$.
2. $y = 10^x$.
3. $y = 2^{-x}$.
4. $y = 3^{-x}$.
5. $x = 2^y$.
6. $y = 2^{x-1}$.

Approximate Numbers

A student who measured the length of the side of a building reported that it was 68 feet, correct to the nearest foot. What information did this result convey? Evidently the result is only approximate because this measurement indicates that the length is nearer 68 than 67 feet or 69 feet. In fact, the actual length is 67.5 or above, but below 68.5, and may have been 67.5, 67.6, 67.7, 67.8, 67.9, 68.0, 68.1, 68.2, 68.3, 68.4. We can see then that the first figure, 6, is *certain*, and the second figure, 8, is approximate or, as we say, *uncertain*, yet has a meaning or significance. It indicates that the actual length in feet is nearer 68 than 67 or 69.

If the student had measured more carefully, correct to the nearest tenth of a foot, and reported the result as 67.8, then the digits 6 and 7 would be *certain* figures but the 8 would be uncertain, and would indicate that the true result is 67.75 or above, but below 67.85. However, a surveyor, making the same measurement even more accurately, may have reported the length as 67.83 correct to the nearest hundredth. Here 6, 7, and 8 are *certain* figures while 3 is uncertain, but has a meaning.

You are now able to understand the following table.

Measurement	Accuracy	Certain figures	Uncertain figure indicates true length	
			is not below	but is below
68 ft.	nearest foot	6	67.5	68.5
68.3 ft.	nearest tenth	6, 8	68.25	68.35
68.74 ft.	nearest hundredth	6, 8, 7	68.735	68.745
3574 ft.	nearest foot	3, 5, 7	3573.5	3574.5
25578.9 ft.	nearest tenth	2, 5, 5, 7, 8	25578.85	25578.95

Thus you will observe that the number of certain figures in a measurement will depend upon the accuracy of the measurement. When the last figure is not zero, it is understood that this last figure is the only uncertain figure in the number.

Significant figures. In the above examples it was stated that there were a number of certain digits and also an uncertain digit which had a meaning or significance. Thus in the measurement 67.8 correct to the nearest tenth, 6 and 7 are the certain figures and 8 the uncertain, yet significant, figure. Therefore the measurement 67.8 correct to the nearest tenth is a number containing three *significant figures.* Similarly 67.83 correct to the nearest hundredth contains four significant figures, and we say that the result is correct to four significant figures.

Again we say that the circumference of the earth is 25,000 miles although it is actually about 24,875 miles. It is customary to round off a large number by replacing some of its figures by zeros. Evidently in 25,000, 2 is the only certain figure and 5 is the only other significant figure, *i.e.*, the number has only two significant figures. The three zeros are not significant. They are used to indicate that the 25 is in the thousands' place and not simply 25 units. Generally zeros are not significant figures. When, however, they occur in the middle of a number, that is, between the figures 1 to 9, they are significant. For example, when we say $1003, we do not mean about $1000, for in that case we would not have added the 3. Consequently the zeros in $1003 are significant.

To say that the circumference of the earth is 25,000 miles estimated to the nearest thousand miles indicates that the three zeros are not significant. If, however, the measurement, 25,000 miles, were correct to the nearest mile, the three zeros would be significant, and this number therefore would contain five significant figures.

If it were required to express the measurement 13.97 correct to the nearest tenth, the result would be written 14.0, and since this zero records the fact that there are no tenths, it is significant. On the other hand, if a student in a chemistry laboratory weighed a substance and reported the weight .05 gram, the zero merely serves to fix the decimal point and is not significant. .05 contains one significant figure, the 5. If we consider the numbers 30560, 3056, 305.6, 30.56, 3.056, .3056, .03056, .003056, etc., we note that all have the same four digits, but differ only in the position of the decimal point. The significant figures in each of these numbers are 3, 0, 5, and 6 in that order.

The number of significant figures is independent of the position of the decimal point. The zero or zeros are not significant if they do not represent an observed or calculated value, but serve merely to place the decimal point.

Thus we see that zero is sometimes significant and sometimes not significant, depending on its use. (1) Zero is significant when it occurs between the figures 1 to 9, or when it indicates the degree of accuracy of the measurement. (2) Zero is not significant when it is used simply to locate the decimal point.

EXERCISES

Express each of the following numbers correct to one less significant figure:

1. 36.92. *Solution:* 36.92 = 36.9.

2. 270.7. *Solution:* 270.7 = 271.

3. 21.3.	**6.** 7.6.	**9.** 10.7.	**12.** 17.98.
4. 19.38.	**7.** 372.5.	**10.** 10.3.	**13.** 35,967.
5. 4.025.	**8.** 216.07.	**11.** 3492.	**14.** .4208.

Copy and underscore the significant zeros in each of the following:

15. 18.09. *Solution:* $18.\underline{0}9$. **16.** 0.078. *Solution:* 0.078.

17. 270.3.	**22.** 8.07.	**27.** 5.082.	**32.** 76.07.
18. 2080.	**23.** 0.783.	**28.** .5082.	**33.** .760.
19. 30.4.	**24.** 18.90.	**29.** .0508.	**34.** .0760.
20. 0.085.	**25.** .0068.	**30.** 250.	**35.** 0.0760.
21. 3700.7.	**26.** .0703.	**31.** 17.03.	**36.** .3070.

Express each of the following to three significant figures:

37. 2732. *Solution:* 2730. **38.** 1798. *Solution:* 1800.

39. 7857.	**44.** 78.067.	**49.** 18.092.	**54.** 10101.
40. 78573.	**45.** 93.067.	**50.** 34902.	**55.** 47.953.
41. 785.7.	**46.** 36.053.	**51.** 78.047.	**56.** 72.48.
42. 785.73.	**47.** 18.96.	**52.** 21870.	**57.** 172.48.
43. 6666.	**48.** 18.92.	**53.** 6.399.	**58.** .9999.

Significant figures and accuracy in computation. Suppose we are required to find the area of a rectangle 8.3 in. by 4.7 in. Computed in the usual way, the area is 39.01 sq. in.

However, if the numbers 8.3 and 4.7 were obtained by actual measurement correct to the nearest tenth, they are approximate numbers. The true length is 8.25 or more, but less than 8.35 and the true width is 4.65 or more, but less than 4.75. Therefore the smallest possible area is 38.3625, obtained by multiplying 8.25 and 4.65, and the largest possible area is 39.5316, obtained by multiplying 8.34 and 4.74. Hence the true area lies somewhere between 38.3625 and 39.5316, and thus is not necessarily exactly 39.01.

If we actually multiply 8.3 by 4.7 and underline the uncertain figures, we see that the area is more correctly estimated as 39, admitting also that this is an approximate answer too. Note that 8.3, 4.7, and 39 are numbers each of which has two significant figures.

$$\begin{array}{r} 8.\underline{3} \\ \underline{4.\underline{7}} \\ 58\underline{1} \\ \underline{33\underline{2}} \\ 3\underline{9}.\underline{01} \end{array}$$

Again suppose we wished to find the area of a rectangle whose length and width are 8.6 and .7 respectively. By multiplying length and width the area is 6.02. However, if the numbers 8.6 and .7 were obtained by actual measurement, correct to the nearest tenth, then the true length is 8.55 or more, but less than 8.65 and the true width is .65 or more, but less than .75. Therefore the smallest possible area is 5.5575, obtained by multiplying 8.55 and .65, and the largest possible area 6.3936, obtained by multiplying 8.64 and .74. Hence the true area lies between 5.5575 and 6.3936 and thus it is clear that the area is not exactly 6.02, but is more correctly estimated as 6, admitting that 6 also is an approximate number.

Note that 8.6 is a number of two significant figures, while .7, and the approximate answer 6, are numbers of only one significant figure.

Therefore, in operations involving approximate numbers, *Express the result to as many significant figures as there are in that one of the given numbers which has the smallest number of significant figures.*

Illustrative example. Multiply 7.8 by 3.08.

Analysis

Since 7.8 has *two* significant figures and 3.08 has *three* significant figures, the result will contain only *two* significant figures. Since 3.08 has three significant figures, our work will be shortened if we first change it to 3.1, that is, correct to *two* significant figures.

Solution

$$7.8 \times 3.1 = 24.18, \text{ which is 24 approximately.}$$

EXERCISES

Multiply each of the following and use the above rule to determine the degree of accuracy of the result.

1. 3.5×5.06.
2. 5.08×3.2.
3. 7.2×5.33.
4. 8.3×6.86.
5. 37.92×7.03.
6. 25.67×4.33.
7. 35.72×3.30.
8. 152.89×7.2.

Approximate measurements and tables. All lines and all related angles must be measured with the same degree of accuracy. If lines are measured to 3 significant figures, the same degree of accuracy requires that the angles be measured to the nearest tenth of a degree; but if the lines are measured to 4 significant figures, the angles should be measured to the nearest hundredth of a degree, in practice usually to the nearest minute.

In using tables it is important then to be consistent with the measurements of the lines and angles. If the lines and angles contain 3 significant figures, it is better to use a three-place table; if 4 significant figures, a four-place table, etc.

The Extension of the Exponent

We have seen that Descartes' use of the Hindu Arabic numerals as exponents was a tremendous step in advance in mathematical notation and symbolism. This symbolism, in addition, opened the way for the use of both negative and fractional numbers as exponents.

The definite explanation of the theory concerning the use of fractional and negative exponents is attributed to John Wallis (1655) and, as might be expected, led to the discovery and solution of many new problems in various fields of mathematics.

CUMULATIVE REVIEW

Chapters X, XI, and XII

1. Which of these statements are true? Which are false?

(*a*) $\sqrt{a^2 + b^2} = a + b$.

(*b*) $\dfrac{1}{\sqrt[3]{4}} = \dfrac{\sqrt[3]{4}}{4}$.

(*c*) $\sin(x - y) = -\sin(y - x)$.

(*d*) $\cos 2A = 2\cos A$ for all values of angle A.

(*e*) If $B = -90°$, then $\cos \frac{1}{2} B = -\frac{1}{2}\sqrt{2}$.

(*f*) $a^{\frac{5}{4}} = \sqrt[4]{a^5}$.

(*g*) $\sqrt{2a^2b} = ab^{\frac{1}{2}}$.

(*h*) $(\frac{8}{27})^{\frac{2}{3}} = \frac{4}{9}$.

(*i*) $(a + b)^0 = 2$.

(*j*) $2 + 2^{-1} = \frac{5}{2}$.

2. Complete each of the following statements:

Expressed in terms of i:

(*a*) $\sqrt{-8a} = ?$

(*b*) $\dfrac{2\sqrt{-8}}{8\sqrt{-2}} = ?$

(*c*) If A is an angle of a triangle such that $\tan A = -\sqrt{3}$, then the value of $\sin\left(\dfrac{3\pi}{2} - A\right)$ is ? .

(*d*) The expression $\dfrac{\cos z}{2} - 1$ is always ? (positive or negative).

(*e*) The smallest positive value of x which satisfies the equation $\cos \frac{1}{2} x = \frac{1}{2}$ is ? degrees.

(*f*) When $a = 2$, the value of $a + 3^0$ is ? .

(*g*) The fraction $\dfrac{x^{-1}\sqrt{2\,m}}{y^2 z}$ may be written without a radical sign and without a denominator as ? .

(*h*) The value of $\dfrac{3 + 3^{-1}}{3}$ is ? .

(*i*) If $2^{2x} = 2^{x+3}$, then $x =$? .

(*j*) The value of $\sqrt[3]{\left(\dfrac{a^{-1}}{b^{-1}}\right)^2} \times \sqrt[3]{\left(\dfrac{a}{b}\right)^2}$ is ? .

3. Express the value of $(\sqrt{-5} - 3)^2$ in terms of i.

4. Solve the equation $\tan x = \sqrt{3 \tan x - 2}$ for two positive values of x less than 90°.

5. Given that $\cos A = \frac{24}{25}$ and A is in Q IV and $\cot B = -\frac{3}{4}$ and B is in Q II, find the value of $\tan (A - B)$.

6. Prove: $\dfrac{\cos (x + y)}{\cos (x - y)} = \dfrac{1 - \tan x \tan y}{1 + \tan x \tan y}$.

7. Solve $\csc y \cos 2\,y + 1 = \csc y$ for all values of y between 0° and 360° and check.

8. Find the value of $4^0 - .04^{\frac{1}{2}} + 81^{-\frac{1}{4}} - (\frac{1}{9})^{\frac{1}{2}} + (\frac{1}{4})^{-\frac{1}{2}}$.

9. (*a*) Multiply $3\,x^2 - 2\,x + 5$ by $2\,x^{-2} + 3\,x^{-1} - 6$.

(*b*) Write the result in descending powers of x.

(*c*) Write the result in (*b*), using positive exponents only.

10. First change $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[6]{7}$ to the same root index, then arrange in descending order of magnitude.

11. Making the appropriate contractions, find the product of 43.71 inches and 5.3 inches.

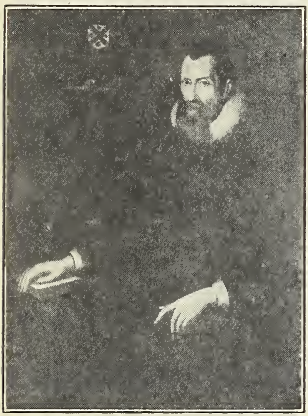
12. Solve for x and check: $5(\frac{1}{2})^{x+1} = \frac{5}{8}$.

CHAPTER XIII. THE MEANING AND USE OF LOGARITHMS

It is no exaggeration to say that the invention of logarithms "by shortening the labors doubled the life of the astronomer." — F. Cajori.

Finding Logarithms

The invention of the steam engine has saved much time in transportation. The invention of the telephone has saved much time in communication. The invention of the adding machine and cash register has reduced the time needed for calculations incidental to business. Likewise, in engineering, astronomy, surveying, insurance, and the like, the tremendous amount of time which would ordinarily be consumed in computation has been considerably reduced by John Napier's discovery of logarithms in the early part of the seventeenth century.



John Napier of Merchiston

We shall find that this discovery is based upon the general laws of exponents, and that, by means of logarithms, multiplication is reduced to addition; division to subtraction; raising to a power to multiplication; and the extraction of a root to division. By the use of logarithms the amount of work involved in the fundamental processes is thus greatly reduced and chances for error lessened.

The use of tables of exponents. The following table gives certain powers of 2, 3, and 10.

Powers of 2		Powers of 3	Powers of 10
$2^0 = 1$	$2^{11} = 2048$	$3^0 = 1$	$10^0 = 1$
$2^1 = 2$	$2^{12} = 4096$	$3^1 = 3$	$10^1 = 10$
$2^2 = 4$	$2^{13} = 8192$	$3^2 = 9$	$10^2 = 100$
$2^3 = 8$	$2^{14} = 16{,}384$	$3^3 = 27$	$10^3 = 1{,}000$
$2^4 = 16$	$2^{15} = 32{,}768$	$3^4 = 81$	$10^4 = 10{,}000$
$2^5 = 32$	$2^{16} = 65{,}536$	$3^5 = 243$	$10^5 = 100{,}000$
$2^6 = 64$	$2^{17} = 131{,}072$	$3^6 = 729$	$10^6 = 1{,}000{,}000$
$2^7 = 128$	$2^{18} = 262{,}144$	$3^7 = 2187$	$10^7 = 10{,}000{,}000$
$2^8 = 256$	$2^{19} = 524{,}288$	$3^8 = 6561$	$10^8 = 100{,}000{,}000$
$2^9 = 512$	$2^{20} = 1{,}048{,}576$	$3^9 = 19{,}683$	$10^9 = 1{,}000{,}000{,}000$
$2^{10} = 1024$	$2^{21} = 2{,}097{,}152$	$3^{10} = 59{,}049$	$10^{10} = 10{,}000{,}000{,}000$

With the help of this table we are able to perform mentally computations that otherwise would involve a large amount of written work.

Illustrative examples.

Example 1. Multiply 32 by 512.

Solution

From the table we see that $32 = 2^5$ and $512 = 2^9$. Therefore, $32 \times 512 = 2^5 \times 2^9$. But the laws of exponents tell us that $2^5 \times 2^9 = 2^{14}$. Referring again to the table, we learn that $2^{14} = 16{,}384$. Therefore, $32 \times 512 = 16{,}384$.

Example 2. Divide 16,384 by 64.

Solution

$$16{,}384 \div 64 = 2^{14} \div 2^6 = 2^8 = 256.$$

Example 3. Find the value of $(81)^2$.

Solution

$$(81)^2 = (3^4)^2 = 3^8 = 6561.$$

Example 4. Find the value of $\sqrt[3]{32{,}768}$.

Solution

$$\sqrt[3]{32{,}768} = (32{,}768)^{\frac{1}{3}} = (2^{15})^{\frac{1}{3}} = 2^5 = 32.$$

Example 5. Simplify $\dfrac{8192 \times 64}{(256)^2 \times \sqrt[5]{1024}}$.

Solution

$$\frac{8192 \times 64}{(256)^2 \times \sqrt[5]{1024}} = \frac{2^{13} \times 2^6}{(2^8)^2 \times \sqrt[5]{2^{10}}}$$

$$= \frac{2^{19}}{2^{16} \times 2^2} = 2.$$

EXERCISES

With the help of the table perform the following indicated operations:

1. 16×128.
2. 256×32.
3. 32×1024.
4. 9×6561.
5. 81×243.
6. $2048 \div 64$.
7. $8192 \div 256$.
8. $32{,}768 \div 512$.
9. $6561 \div 27$.
10. $6561 \div 243$.
11. $(32)^2$.
12. $(128)^3$.
13. $(32)^3$.
14. $(81)^2$.
15. $(9)^3$.
16. $\sqrt{1024}$.
17. $\sqrt{16{,}384}$.
18. $\sqrt[3]{4096}$.
19. $\sqrt{6561}$.
20. $\sqrt[3]{729}$.
21. $\sqrt[4]{4096}$.
22. $\sqrt[4]{256}$.
23. $\sqrt[4]{6561}$.
24. $\sqrt[6]{729}$.
25. $9 \times 27 \times 243$.
26. $2187 \times \sqrt{729}$.
27. $\dfrac{729 \times 2187}{59{,}049}$.
28. $\dfrac{64 \times 256 \times 8}{16{,}384}$.
29. $\dfrac{2048 \times (128)^3 \times 64}{(256)^3 \times \sqrt[3]{32{,}768}}$.
30. $\sqrt[3]{512} \times \sqrt{16{,}384}$.
31. What exponent must 2 have to give 256? 8? 1? 4096?
32. What exponent must 10 have to give 10? 100? 1? 10,000? 1,000,000?

The meaning of a logarithm. As $2^4 = 16$, the exponent which 2 must have to give 16 is 4; as $4^2 = 16$, the exponent which 4 must have to give 16 is 2. In these two examples and in the table of powers of numbers, the exponents are given a special name — *logarithms* (*log*). However, since logarithms are really exponents, it is helpful when working with logarithms to think of them as exponents.

In $2^4 = 16$, the exponent 4 is called the *logarithm;* the number 2 is called the *base;* and the number 16 is called the *resulting number.*

As you see, 16 is the resulting number when the base 2 has the exponent or logarithm 4. In the equation $2^4 = 16$, 4 is the logarithm of 16 to the base 2. In general, in the equation $b^x = N$, b is the base, N is the resulting number, and the exponent x is the logarithm. In reading this equation we say that x is the logarithm of N to the base b, or $\log_b N = x$.

The logarithm of a number N is the exponent that indicates the power to which another number (called the base) must be raised to give that number N.

The form $b^x = N$ is called the *exponential form;* the form $\log_b N = x$ is called the *logarithmic form.*

Illustrative examples.

Example 1. Write $\log_3 81 = 4$ in the exponential form.

Solution

$$\log_3 81 = 4 \text{ in the exponential form is } 3^4 = 81.$$

Example 2. Write $2^5 = 32$ in the logarithmic form.

Solution

$$2^5 = 32 \text{ in logarithmic form is } \log_2 32 = 5.$$

Example 3. Determine the logarithm of 128 to the base 2.

Solution

Let $\log_2 128 = x$, then $2^x = 128$.

But from the table, $2^7 = 128$. $\therefore 2^x = 2^7$ and $x = 7$.

EXERCISES

Write in logarithmic form:

1. $2^8 = 256$.
2. $3^2 = 9$.
3. $10^2 = 100$.
4. $6^2 = 36$.
5. $10^4 = 10{,}000$.
6. $4^3 = 64$.
7. $5^3 = 125$.
8. $2^0 = 1$.
9. $10^{-2} = .01$.
10. $3^{-2} = \frac{1}{9}$.
11. $4^{-2} = \frac{1}{16}$.
12. $(\sqrt{3})^2 = 3$.

Write each of the following in the exponential form:

13. $\log_5 25 = 2$.
14. $\log_8 64 = 2$.
15. $\log_{10} 100 = 2$.
16. $\log_{10} 10 = 1$.
17. $\log_{10} .01 = -2$.
18. $\log_{10} .001 = -3$.
19. $\log_2 16 = 4$.
20. $\log_{\sqrt{2}} 2 = 2$
21. $\log_3 27 = 3$.
22. $\log_3 \frac{1}{9} = -2$.
23. $\log_{10} .0001 = -4$.
24. $\log_b N = x$.
25. $\log_4 8 = \frac{3}{2}$.
26. $\log_8 4 = .67$.
27. $\log_8 16 = 1.33$.

Determine each of the following logarithms:

28. $\log_{10} 100$.
29. $\log_2 128$.
30. $\log_3 27$.
31. $\log_2 16$.
32. $\log_3 729$.
33. $\log_{10} .1$.
34. $\log_2 \frac{1}{16}$.
35. $\log_5 1$.
36. $\log_6 36$.
37. $\log_{10} 10{,}000$.
38. $\log_{\sqrt{2}} 4$.
39. $\log_{\sqrt{5}} 5$.
40. $\log_4 32$.
41. $\log_{27} 9$.
42. $\log_{32} 64$.

Something to think about.

1. What is the logarithm of 1 to any base?

2. What is the logarithm of a number, having that same number as the base?

3. If any positive base, as 10, is raised to a positive exponent, is the result positive or negative?

4. If any positive base, as 10, is raised to a negative exponent, is the result positive or negative?

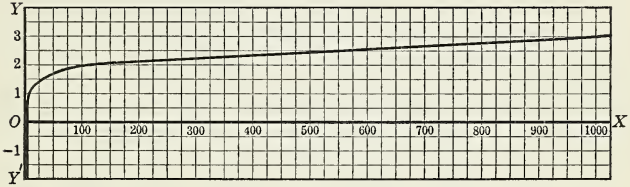
5. If any positive base, as 10, is raised to any fractional exponent, and considering principal roots only, can we get a negative result?

6. If the base is positive, can a negative number have a real logarithm?

The graph of $y = \log_{10} x$. Expressing $y = \log_{10} x$ in the exponential form, $x = 10^y$. Assigning values to y and computing the corresponding values for x, we have the table.

If $y =$	3	2	1	0	-1	-2	-3
Then $x =$	1000	100	10	1	.1	.01	.001

Plotting these points, we have the graph below.



Some observations from the graph.

1. A negative number has no real logarithm if the base is positive. Explain from the graph.

2. The logarithm of a number between 0 and 1 is negative.

3. The logarithm of 1 is zero.

4. The logarithm of a number increases as the number increases.

5. Beyond $x = 100$ the graph becomes very nearly a straight line and thus an increase in the value of x causes, very closely, a proportional increase in the value of y. (See part of graph between $x = 500$ and $x = 600$.)

Common logarithms. Thus far you have observed that any number may be regarded as the base of a logarithm. However, 10 is the base most commonly used, because our decimal system of numbers makes 10 most convenient for tabular purposes. Logarithms with the base 10 are called *common logarithms* and we shall learn how to find the common logarithm of any number. Since the number 10 will from now on be used as the base in our work with logarithms, we shall omit the base in writing the logarithmic form, but with the understanding that it is 10. Thus log 250 means $\log_{10} 250$.

Characteristic and mantissa of a logarithm. If all the numbers in our number system were exact powers of 10, the logarithms of such numbers would be integers. Why? However, most numbers are not powers of 10 and therefore our problem is to find a method of determining the logarithm of any number. Study carefully the following table of powers of 10 and the corresponding logarithmic forms and answer the questions following it.

$10^0 = 1$	$\therefore \log 1 = 0.$	$10^0 = 1$	$\therefore \log 1 = 0.$
$10^1 = 10$	$\therefore \log 10 = 1.$	$10^{-1} = .1$	$\therefore \log .1 = -1.$
$10^2 = 100$	$\therefore \log 100 = 2.$	$10^{-2} = .01$	$\therefore \log .01 = -2.$
$10^3 = 1,000$	$\therefore \log 1,000 = 3.$	$10^{-3} = .001$	$\therefore \log .001 = -3.$
$10^4 = 10,000$	$\therefore \log 10,000 = 4.$	$10^{-4} = .0001$	$\therefore \log .0001 = -4.$

1. Since 7 is between 1 and 10, between what numbers does its logarithm lie?

2. Between what numbers does log 95 lie? log 367? log 6874?

3. Since .4 is between 1 and .1, between what numbers does its logarithm lie?

4. Between what numbers does log .08 lie? log .002? log .0006?

You can see that the logarithm of any number (x) between 100 and 1000 lies between 2 and 3; and is 2 + a decimal.

Thus: $\log 100 = 2.$
$\log x = 2 + \text{decimal}.$
$\log 1000 = 3.$

A logarithm, then, consists of two parts, an integral and a decimal part. The integral part is called the *characteristic* of the logarithm; the decimal part is called the *mantissa.*

The table shows that the characteristic may be either positive or negative, but for convenience we always use a positive mantissa.

The logarithm of any number between 100 and 1000 is 2 plus a decimal. Therefore, to determine the logarithm of such a number we must find two parts, the characteristic and the mantissa. The former we shall learn to find by inspection, and the latter we shall get from a table.

Determining the characteristic.

Number	.001	.01	.1	1	10	100	1000	10,000
Logarithm	− 3	− 2	− 1	0	1	2	3	4

From this table, verify the following characteristics:

1. $\log 1 = 0.$

2. $\log 75 = 1.^+.$

3. $\log 75.075 = 1.^+.$

4. $\log 157.589 = 2.^+.$

5. $\log 1753.89 = 3.^+.$

6. $\log 9.89 = 0.^+.$

7. $\log .859 = -1.^+.$

8. $\log .06 = -2.^+.$

9. $\log .0603 = -2.^+.$

10. $\log .002 = -3.^+.$

11. $\log. 00207 = -3.^+.$

12. $\log 7 = 0.^+.$

13. $\log 70 = 1.^+.$

14. $\log 700 = 2.^+.$

15. $\log 7000 = 3.^+.$

16. $\log 3 = 0.^+.$

17. $\log .3 = -1.^+.$

18. $\log .03 = -2.^+.$

When there is just one significant figure to the left of the decimal point, as in log 7 (Ex. 12), the characteristic is 0. When the decimal point is moved one place to the right, as in log 70 (Ex. 13), the characteristic becomes 1. When it is moved another place to the right, as in log 700 (Ex. 14), the characteristic becomes 2. Notice that for each place the decimal point is moved to the right, the characteristic is increased by 1.

In log 3 (Ex. 16) there is but one significant figure to the left of the decimal point and the characteristic is again 0. When the decimal point is moved one place to the left, as in log .3 (Ex. 17), the characteristic becomes − 1. When it is moved another place to the left, as in log .03 (Ex. 18), the characteristic becomes − 2. Notice that for every place that the decimal point is moved to the left, the characteristic is decreased by 1.

To find the characteristic of a number, first think of the decimal point as being after the first significant figure of the number. The number then would lie between 1 and 10. Then the characteristic will equal numerically the number of places from this imaginary decimal point to the real decimal point, and will be positive if you count from the imaginary decimal point to the right, and negative if you count to the left.

Illustrative examples.

Example 1. Find the characteristic of the logarithm of 473.8.

Solution

If the decimal point were after the 4, the characteristic would be 0. Since it is two places farther to the right, the characteristic is 2.

Example 2. Find the characteristic of the logarithm of .0008742.

Solution

If the decimal point were after the 8, the characteristic would be 0. Since it is four places farther to the left, the characteristic is − 4.

Example 3. Find the characteristic of the logarithm of 5.347.

Solution

If the decimal point were after the 5, the characteristic would be 0. Since it is there, the characteristic is 0.

Determining the decimal point from the characteristic. If the characteristic of the logarithm of a number is given, this same rule reversed can be used to locate the decimal point in the number.

Example 4. The sequence of the digits of a number is 14293. If the characteristic of the logarithm of the unknown number is 2, locate its decimal point.

Solution

If the characteristic were 0, the decimal point would be after the 1. Since it is 2, the decimal point must be two places farther to the right or after the 2. The number, then, is 142.93.

Example 5. The sequence of the digits of a number is 3471. If the characteristic of the logarithm of the unknown number is -3, locate the decimal point.

Solution

If the characteristic were 0, the decimal point would be after the 3. Since the characteristic is -3, the decimal point must be three places farther to the left. The number, then, is .003471.

Writing the negative characteristic. From the previous tables we can readily see that log .0859 is $-2 +$ a mantissa. For the present, let us assume this mantissa to be .9340.

Thus, log. $.0859 = -2 + .9340$, which is really equivalent to -1.0660, a negative characteristic and a negative mantissa. But for practical computations it is more convenient to use a positive mantissa. We must therefore find another way of writing log .0859 so that the mantissa will be positive. There are two ways:

(1) $\log .0859 = \bar{2}.9340$. The minus sign above the 2 indicates that the characteristic alone is negative; the mantissa is positive.

(2) $\log .0859 = 8.9340 - 10$. This form is obtained by adding $10 - 10$, or zero, to the logarithm, thus:

$$\begin{array}{l} -2 + .9340 \\ \underline{10 \qquad\quad - 10} \\ 8 + .9340 - 10 \\ \quad = 8.9340 - 10. \end{array}$$

Notice that the mantissa is positive and that we have not altered the value of the logarithm by adding and subtracting 10, because $10 - 10 = 0$. This form is the one more adaptable to logarithmic computation.

EXERCISES

State the characteristic of:

1. log 85.
2. log 72.
3. log 700.
4. log 3456.
5. log 25.89.
6. log 5765.
7. log .05.
8. log .0508.
9. log .502.
10. log .0502.
11. log .000205.
12. log .059.
13. log .089.
14. log .0056.
15. log .0005.

Write a number, the characteristic of whose logarithm is:

16. 1.
17. 3.
18. 0.
19. -1.
20. 2.
21. -3.
22. -2.
23. -4.

Determining the mantissa. The mantissas of numbers are found in Table I on pages 572 and 573. From this table let us find the logarithm of 258. The characteristic of this number is 2. Why? To find the mantissa, turn to the table of mantissas and look down the left-hand column headed N until you come to 25, the first two figures of the number 258. Then look horizontally across the table until you come to the column headed by 8, the third figure. There you will find 4116, which is the mantissa of 258. The decimal point before the 4116 is omitted in the table to save space. Log 258, then, is 2.4116.

Now since the logarithm of every number is composed of a characteristic and a mantissa, it would seem that we ought to have volumes of pages in the table of mantissas instead of only two pages, because there are innumerable numbers. The reason that only two pages are needed will be shown in the following application of the laws of exponents to the making of a table of logarithms of numbers.

The statement

$$\log 258 = 2.4116 \text{ means } 10^{2.4116} = 258. \qquad (1)$$

If we multiply both sides of (1) by 10, we have

$$10^{2.4116} \times 10^{1} = 258 \times 10.$$

$$\therefore\ 10^{3.4116} = 2580, \qquad (2)$$

or $$\log 2580 = 3.4116.$$

Multiplying (2) by 10,

$$10^{4.4116} = 25800,$$

or $$\log 25800 = 4.4116.$$

Dividing (1) by 10,

$$10^{1.4116} = 25.8,$$

or $$\log 25.8 = 1.4116.$$

Arranging these results and others obtained by repeated divisions by 10 in a table, we have:

$$\begin{aligned}
10^{4.4116} &= 25800, &&\text{or } \log 25800 = 4.4116.\\
10^{3.4116} &= 2580, &&\text{or } \log 2580 = 3.4116.\\
10^{2.4116} &= 258, &&\text{or } \log 258 = 2.4116.\\
10^{1.4116} &= 25.8, &&\text{or } \log 25.8 = 1.4116.\\
10^{0.4116} &= 2.58, &&\text{or } \log 2.58 = 0.4116.\\
10^{9.4116-10} &= .258, &&\text{or } \log .258 = 9.4116 - 10.\\
10^{8.4116-10} &= .0258, &&\text{or } \log .0258 = 8.4116 - 10.
\end{aligned}$$

What conclusions do these results suggest? (1) The characteristic changes, and its changes are consistent with the rules already stated for finding the characteristic. (2) For the same sequence of digits, the mantissa is the same regardless of the position of the decimal point. That is, the mantissa for 2580 is the same as that for .0258.

In general, numbers containing the same sequence of digits and differing only in the position of the decimal point have the same mantissa. This conclusion shows why we do not need to pay any attention to the decimal point in the number when looking up the mantissa.

Finding the logarithm of a number.

Type A. The number does not contain more than three significant figures.

Illustrative examples.

Example 1. Find the logarithm of 54.7.

Solution

The characteristic is 1. Why?

The mantissa is the same as that of 547. Why?

In the table, look down the column headed N until you come to 54 (the first two digits of 547) and then horizontally across the table to column headed 7, in which you will find .7380.

$$\therefore \log 54.7 = 1.7380.$$

Example 2. Find the logarithm of .00234.

Solution

The characteristic is -3, or $7 - 10$. Why?

The mantissa is the same as that of 234.

Consulting the table, we find the mantissa of 234 is .3692.

$$\therefore \log .00234 = 7.3692 - 10.$$

Example 3. Find the logarithm of .5.

Solution

The characteristic is -1. Why?

The mantissa is the same as that of 50, 500, 5000, etc. Why?

Consulting the table for the mantissa of 500 we find .6990.

$$\therefore \log .5 = 9.6990 - 10.$$

EXERCISES

Using the table of mantissas, verify each of the following:

1. log 74.3 = 1.8710.
2. log 237 = 2.3747.
3. log 359 = 2.5551.
4. log 32 = 1.5051.
5. log 650 = 2.8129.
6. log .872 = 9.9405 − 10.
7. log .0856 = 8.9325 − 10.
8. log 5 = .6990.
9. log .05 = 8.6990 − 10.
10. log .0078 = 7.8921 − 10.

Find the logarithm of each of the following numbers:

11. 173.
12. 259.
13. 3.57.
14. 28.2.
15. 300.
16. 30.
17. 3.
18. .03.
19. .003.
20. 5.3.
21. 620.
22. 62.
23. .0062.
24. 6200.
25. .078.
26. .0053.
27. .0532.
28. .518.
29. 700.
30. .007.

Type B. The number contains more than three significant figures.

Illustrative examples.

Example 1. Find the logarithm of 234.3.

Solution

The characteristic of log 234.3 is 2. Why?

Now that the decimal point has been taken care of, it is necessary to find the mantissa of 2343. Since the first three digits can be located in the table, the mantissa of 2343 must lie between the mantissa of 2340 and 2350. Why? From the table we may write:

$$\begin{array}{rcccl}
 & \textit{Number} & & \textit{Mantissa} & \\
\text{increase in number} = 3 & 2340 & \longrightarrow & .3692 & \\
 & 2343 & \text{increase in number} = 10 & ? & \text{increase in mantissa} = .0019. \\
 & 2350 & \longrightarrow & .3711 &
\end{array}$$

Note that for an increase of 10 in the number there is a corresponding increase of .0019 in the mantissa. The graph of $y = \log x$ (page 421) showed us that an increase in the value of x is accompanied by a very nearly proportional increase in the value of y. Hence, in the example on the preceding page, since an increase of 10 in the number causes an increase of .0019 in the mantissa, an increase of 3 in the number causes an increase of $\frac{3}{10}$ of .0019, or .0006 in the mantissa. Adding .0006 to .3692, the mantissa of 2340, we have .3698 as the mantissa of 2343.

$$\therefore \log 234.3 = 2.3698.$$

The method of computing a number which in value would be between two consecutive numbers in a table is called *interpolation.*

Example 2. Find the logarithm of .056782.

Solution

The characteristic is $\bar{2}$ or 8. − 10.

Since we are using a four-place table, write the given number correct to four significant figures. (A four-place table is a table which gives values correct to four significant figures. In this table of mantissas, three significant figures are given directly; the fourth can be found by interpolation.) Here then we are to find the logarithm of .05678. The number 5678 lies between 5670 and 5680. From the table determine the mantissas for 5670 and 5680 and arrange as follows:

$$\begin{array}{r l c l l}
 & \textit{Number} & & \textit{Mantissa} & \\
\text{increase} = 8 & 5670 & \longrightarrow & .7536 & \\
 & 5678 & & & \\
 & & \text{increase} = 10 & & \text{increase} = .0007. \\
 & 5680 & \longrightarrow & .7543 &
\end{array}$$

Since $\quad \frac{8}{10}$ of $.0007 = .0006.$

Adding, $\quad .0006 + .7536 = .7542.$

$$\therefore \log .056782 = 8.7542 - 10$$

EXERCISES

Find the logarithms of:

1. 2345.	**8.** .07893.	**15.** .00067853.
2. 45.67.	**9.** .1734.	**16.** 247238.
3. .6789.	**10.** 5807.	**17.** 7832.47.
4. 891.2.	**11.** 18765.	**18.** 32.8745.
5. 1.324.	**12.** 765.38.	**19.** .778293.
6. .06532.	**13.** 4.5867.	**20.** .00607236.
7. .0008936.	**14.** .054687.	**21.** 76.243

Antilogarithms. Up to this time we have learned how to determine the logarithm of a given number, such as log 28.7 = 1.4579. Our problem now is to find a number whose logarithm is given. That is, if $\log x = 1.4579$, find x.

We have learned that in any logarithm: (1) the characteristic determines the position of the decimal point; and (2) the mantissa determines the sequence of digits in the number.

Thus, to find the number corresponding to a given logarithm, that is, to find the *antilogarithm* of a number, we must (*a*) determine the sequence of digits in the required number, and (*b*) then decide on the position of the decimal point. The following illustrative examples will indicate the procedure.

Illustrative examples.

Example 1. Find the antilogarithm of 1.5717; that is, find the number whose logarithm is 1.5717.

Solution

Look in the table of mantissas for the mantissa .5717. This number is in the horizontal row corresponding to 37 in the N column and in the column headed 3. Thus .5717 is the mantissa of a number whose digits in order are 373. This we shall call the *key number*.

If the characteristic were 0, the decimal point would be placed after the first 3. Since the characteristic is 1, count one place to the right and place the decimal point after the 7. Thus the required number must be 37.3.

$\therefore$ the antilogarithm of 1.5717 is 37.3.

Example 2. Find the antilogarithm of $8.8041 - 10$.

Solution

Consulting the table of mantissas for the mantissa .8041, we find the key number 637.

Now the characteristic is $8 - 10$ or -2. Therefore, think of a decimal point after the 6 and then count two places to the left. Therefore there must be one zero immediately following the decimal point in the required number.

$\therefore$ the antilogarithm of $8.8041 - 10 = .0637$.

Example 3. Find the antilogarithm of 0.6518.

Solution

Consulting the table of mantissas, we find that the mantissa .6518 is lacking, and therefore we must interpolate. We note that .6518 lies between .6513 and .6522 and the key number corresponding to .6518 is determined as follows:

	Number		*Mantissa*	
	4480	←	.6513	increase = .0005.
increase = 10		increase = .0009	.6518	
	4490	←	.6522	

An increase of .0009 in the mantissa corresponds to an increase of 10 in the number. Therefore an increase of .0005 in the mantissa corresponds to an increase in the number of $\frac{.0005}{.0009} \times 10 = \frac{5}{9} \times 10 = 5.5 = 6$ correct to the nearest integer. Adding 6 to 4480, we obtain the key number 4486.

The characteristic is zero, which indicates that the decimal point belongs after the first significant figure.

$\therefore$ the antilogarithm of 0.6518 is 4.486.

EXERCISES

Find the number corresponding to each of the following logarithms:

1. 0.1790.	**8.** 9.4886 − 10.	**15.** 7.9294 − 10.
2. 1.4116.	**9.** 8.8779 − 10.	**16.** 6.9666 − 10.
3. 2.8420.	**10.** 7.9773 − 10.	**17.** 9.0000 − 10.
4. 3.9047.	**11.** 6.4857 − 10.	**18.** 8.8698 − 10.
5. 4.6053.	**12.** 3.0000.	**19.** 2.8704.
6. 5.4099.	**13.** 9.4425 − 10.	**20.** .9274.
7. .7380.	**14.** 8.8982 − 10.	**21.** 0.5866.

Find the antilogarithm of each of the following:

22. 1.6842.	**28.** .4089.	**34.** 8.8867 − 10.
23. 2.5293.	**29.** 9.5153 − 10.	**35.** 9.8766 − 10.
24. 3.8064.	**30.** 8.3156 − 10.	**36.** 8.5880 − 10.
25. 4.9273.	**31.** 5.8498.	**37.** 7.4875 − 10.
26. 6.8828 − 10.	**32.** .9061.	**38.** 6.4055 − 10.
27. 3.0058.	**33.** 2.4736.	**39.** 3.6110.

Computing with Logarithms

The four " Pillars " of logarithms. Although we are almost ready actually to use our knowledge of logarithms in computations, we must still develop the fundamental principles or laws of logarithms. These are the principles which simplify and shorten numerical calculations.

I. *The logarithm of a product is equal to the sum of the logarithms of its factors;* that is,

$$\log_b (M \times N) = \log_b M + \log_b N.$$

Proof

Let $\log_b M = x$ and $\log_b N = y$. (1)

Then $b^x = M$ and $b^y = N$.

Multiplying the last two equations, we have

$$b^{x+y} = M \times N. \quad (2)$$

Writing (2) in logarithmic form, we have

$$\log_b (M \times N) = x + y. \quad (3)$$

Substituting the values for x and y from (1) into (3),

$$\log_b (M \times N) = \log_b M + \log_b N.$$

Omitting the base, we have in general

$$\log (M \times N) = \log M + \log N.$$

Extending this idea, we have

$$\log (M \times N \times P \times \cdots \text{etc.}) = \log M + \log N + \log P + \cdots \text{etc.}$$

This means that

$$\log (6 \times 3 \times 2) = \log 6 + \log 3 + \log 2.$$

Illustrative examples.

Example 1. Multiply 3.74 by 57.2.

Estimate

If we consider 3.74 as 4 and 57.2 as 57, we see that the product should be somewhere near 4×57 or 228.

Solution

Let $x = 3.74 \times 57.2.$

Since the two sides of this equation are equal, the logarithm of one side must equal the logarithm of the other.

Then $\log x = \log 3.74 + \log 57.2.$

$$\begin{aligned} \log 3.74 &= .5729 \\ \log 57.2 &= \underline{1.7574} \\ \therefore \log x &= 2.3303 \\ \therefore x &= 213.9. \quad \text{(Finding the antilogarithm.)} \end{aligned}$$

By multiplication $3.74 \times 57.2 = 213.928$, to which our answer, obtained by logarithms, is extremely close, considering that we used a four-place table. Again, if the numbers 3.74 and 57.2 are measurements, since each has 3 significant figures, the answer correct to three significant figures is 214, with which our answer agrees.

Example 2. Multiply 35.78 by .002357.

Estimate

$$36 \times .002 = .072.$$

Solution

$$\begin{aligned} \text{Let} \qquad x &= 35.78 \times .002357. \\ \text{Then} \qquad \log x &= \log 35.78 + \log .002357. \\ \log 35.78 &= 1.5537 \\ \log .002357 &= \underline{7.3724} - 10 \\ \log x &= 8.9261 - 10 \\ \therefore x &= .08436. \end{aligned}$$

EXERCISES

Find by logarithms the indicated products:

1. 27.2×3.82.
2. 273×5.83.
3. $.555 \times 539$.
4. $32.5 \times .726$.
5. $.345 \times .0679$.
6. $.00562 \times .821$.
7. 35.8×17.58.
8. 4.86×369.2.
9. $.681 \times 7.6543$.
10. 34.82×17.85.
11. $.5381 \times 72.56$.
12. 12345×678921.
13. $.53 \times .72 \times 7695$.
14. $27.8 \times 278 \times 6342$.
15. $.000865 \times 370 \times 53.22$.
16. $.5 \times .000003 \times 376.54$.
17. $7619 \times .054 \times 12.358$.
18. $32.24 \times 76.41 \times .000763$.

II. *The logarithm of a quotient or fraction is equal to the logarithm of the numerator minus the logarithm of the denominator;* that is,

$$\log_b \frac{M}{N} = \log_b M - \log_b N.$$

Proof

Let $\log_b M = x$ and $\log_b N = y$. (1)

Then $b^x = M$ and $b^y = N$.

Dividing the last two equations, $b^{x-y} = \frac{M}{N}$. (2)

Writing (2) in logarithmic form, $\log_b \frac{M}{N} = x - y$. (3)

Substituting the values for x and y from (1) into (3), we have

$$\log_b \frac{M}{N} = \log_b M - \log_b N.$$

Omitting the base, we have in general,

$$\log \frac{M}{N} = \log M - \log N.$$

That is, $\log \frac{3}{5} = \log 3 - \log 5$ and $\log \frac{2}{3} = \log 2 - \log 3$.

Illustrative examples.

Example 1. Find the value of $572 \div 37.8$.

Estimate

$\frac{600}{40}$ or 15.

Solution

Let $x = \frac{572}{37.8}$.

Then

$$\begin{aligned} \log x &= \log 572 - \log 37.8. \\ \log 572 &= 2.7574 \\ \log 37.8 &= \underline{1.5775} \\ \log x &= 1.1799 \\ x &= 15.13. \end{aligned}$$

If the given numbers are measurements, since each has 3 significant figures, the answer, correct to the same degree of accuracy, is 15.1.

Example 2. Find the value of $38.46 \div .006825$.

Estimate

$$39 \div .007 \text{ or } 5500.$$

Solution

Let $$x = \frac{38.46}{.006825}.$$

Then
$$\begin{aligned} \log x &= \log 38.46 - \log .006825. \\ \log 38.46 &= 1.5850 \\ \log .006825 &= \underline{7.8341 - 10} \end{aligned}$$

Here you observe that the subtrahend is larger than the minuend. We therefore add 10 to the minuend and then subtract 10 from the minuend to avoid complications in the subtraction. Thus,

$$\begin{aligned} \log 38.46 &= 11.5850 - 10 \\ \log .006825 &= \underline{7.8341 - 10} \\ \log x &= 3.7509 \\ x &= 5635. \end{aligned}$$

EXERCISES

Find by logarithms the value of each of the following fractions:

1. $\frac{37.5}{1.39}$.
2. $\frac{47.8}{2.61}$.
3. $\frac{474}{312}$.
4. $\frac{742}{4.38}$.
5. $\frac{.816}{24.7}$.
6. $\frac{45.8}{1.733}$.
7. $\frac{.361}{2348}$.
8. $\frac{1.876}{24.13}$.
9. $\frac{.5369}{75.81}$.
10. $\frac{.3675}{.005928}$.
11. $\frac{3.982}{.05671}$.
12. $\frac{123.456}{.789123}$.
13. $\frac{476.324}{.32756}$.
14. $\frac{481\,005}{.000073}$.
15. $\frac{6.874}{.00004656}$.

III. *The logarithm of the power of a number is equal to the logarithm of the number multiplied by the exponent of the power;* that is,

$$\log_b M^p = p \log_b M.$$

Proof

Let $\qquad \log_b M = x. \qquad (1)$

Then $\qquad b^x = M.$

Raising each number to the power p,

$$(b^x)^p = (M)^p.$$

$$b^{px} = M^p. \qquad (2)$$

Writing (2) in logarithmic form,

$$\log_b M^p = px. \qquad (3)$$

Substituting the value of x from (1) into (3),

$$\log_b M^p = p \log_b M.$$

Omitting the base, we have in general,

$$\log M^p = p \log M.$$

That is, $\log 2^3 = 3 \log 2$; $\log 3^{20} = 20 \log 3$.

Illustrative example. Find the value of $(.0234)^3$.

Estimate

$.02 \times .02 \times .02$ or .000008. An estimate such as this tells us the approximate position of the decimal point and prevents our giving an absurd answer.

Solution

Let $\qquad x = (.0234)^3.$

Then $\qquad \log x = 3 \log .0234.$

$$\log .0234 = 8.3692 - 10.$$

$$\therefore 3 \log .0234 = 25.1076 - 30.$$

$$\therefore \log x = 5.1076 - 10.$$

$$x = .00001281.$$

Assuming .0234 is a measurement, since it contains 3 significant figures, the answer, correct to the same degree of accuracy, is .0000128.

EXERCISES

Find by logarithms the value of each of the following:

1. $(3.5)^3$.	**5.** $(1.03)^{12}$.	**9.** $(565.9)^3$.
2. $(18.9)^4$.	**6.** $(1.045)^{20}$.	**10.** $(4.568)^2$.
3. $(.765)^4$.	**7.** $(.00215)^4$.	**11.** $(.3682)^3$.
4. $(1.05)^5$.	**8.** $(34.78)^4$.	**12.** $(4.27813)^2$.

IV. *The logarithm of a root of a number is equal to the logarithm of the number divided by the root index;* that is,

$$\log_b \sqrt[p]{M} = \frac{\log_b M}{p}.$$

Proof

Let $\log_b M = x.$ (1)

Then $b^x = M.$

Extracting the pth root of each member, we have

$$\sqrt[p]{b^x} = \sqrt[p]{M}, \text{ or } b^{\frac{x}{p}} = \sqrt[p]{M}. \quad (2)$$

Writing (2) in logarithmic form, we have

$$\log_b \sqrt[p]{M} = \frac{x}{p}. \quad (3)$$

Substituting the value of x from (1) in (3), we have

$$\log_b \sqrt[p]{M} = \frac{\log_b M}{p}.$$

Omitting the base, we have in general,

$$\log \sqrt[p]{M} = \frac{\log M}{p} = \frac{1}{p} \log M.$$

That is, $\log \sqrt[3]{4} = \frac{1}{3} \log 4$; $\log \sqrt{30} = \frac{1}{2} \log 30$.

Illustrative examples.

Example 1. Find the value of $\sqrt[3]{17.8}$.

Estimate

Since $2^3 = 8$ and $3^3 = 27$, the answer lies between 2 and 3.

Solution

Let
$$x = \sqrt[3]{17.8}.$$
$$\therefore \log x = \tfrac{1}{3} \log 17.8.$$
$$\log 17.8 = 1.2504.$$
$$\tfrac{1}{3} \log 17.8 = .4168.$$
$$\therefore \log x = .4168.$$
$$\therefore x = 2.611.$$

Assuming 17.8 is a measurement, since it contains 3 significant figures, the answer, correct to the same degree of accuracy, is 2.61.

Example 2. Find the value of $\sqrt[3]{.1632}$.

Estimate

Since $(.5)^3 = .125$, the answer is slightly larger than .5.

Solution

Let
$$x = \sqrt[3]{.1632}.$$
$$\therefore \log x = \tfrac{1}{3} \log .1632.$$
$$\log .1632 = 9.2127 - 10.$$

Now if this logarithm were divided by 3, the quotient would be $3.0709 - 3\frac{1}{3}$. But $3 - 3\frac{1}{3}$ is supposed to indicate either the number of digits or the number of zeros, that is, it determines the decimal point. Since a fractional number of either is impossible, we must write the original logarithm so that the number at the end is an exact multiple of the divisor. Thus we may say, $\log .1632 = 9.2127 - 10$ or $29.2127 - 30$.

$$\therefore \tfrac{1}{3} \log .1632 = 3\overline{)29.2127 - 30}$$
$$\log x = \quad 9.7376 - 10$$
$$\therefore x = .5465.$$

EXERCISES

Find by logarithms the value of each of the indicated roots:

1. $\sqrt{565}$.
2. $\sqrt[4]{27.8}$.
3. $\sqrt[4]{12.8}$.
4. $\sqrt[5]{57.2}$.
5. $\sqrt[3]{.509}$.
6. $\sqrt{2789}$.

7. $\sqrt[4]{4.325}$.
8. $\sqrt[3]{.45678}$.
9. $\sqrt{.002165}$.
10. $\sqrt[3]{.04762}$.
11. $\sqrt[4]{.001395}$.
12. $389^{\frac{1}{3}}$.
13. $72.96^{\frac{1}{10}}$.
14. $1.685^{\frac{1}{9}}$.
15. $7.432^{\frac{1}{5}}$.
16. $.6791^{\frac{2}{5}}$.
17. $94.16^{11.5}$.
18. $3.157^{2.6}$.

Combined operations. By applications of the laws of logarithms, much time and labor may be saved in complicated computations, as the following examples will illustrate.

Illustrative examples.

Example 1. Find the value of $\dfrac{(.846)^2 \times \sqrt[3]{18.7}}{3.42}$.

Estimate

We may regard $(.846)^2$ as .6; $\sqrt[3]{18.7}$ as 2.5; and 3.42 as 3.

$\therefore$ estimated value is $\dfrac{.6 \times 2.5}{3} = \dfrac{1.5}{3} = .5$.

Solution

Let $$x = \frac{(.846)^2 \times \sqrt[3]{18.7}}{3.42}.$$

$$\begin{aligned}
\therefore \log x &= 2 \log .846 + \tfrac{1}{3} \log 18.7 - \log 3.42. \\
\log .846 &= 9.9274 - 10 \\
\log 18.7 &= 1.2718 \\
2 \log .846 &= 19.8548 - 20 \\
\tfrac{1}{3} \log 18.7 &= \underline{.4239} \\
& 20.2787 - 20 \\
\log 3.42 &= \underline{.5340} \\
\log x &= 19.7447 - 20 \\
\log x &= 9.7447 - 10. \\
\therefore x &= .5555.
\end{aligned}$$

Assuming the given numbers are the results of measurements, since no number contains less than 3 significant figures, the answer correct to the same degree of accuracy is .556.

Example 2. Find by logarithms the value of

$$\frac{-.75 \times \sqrt[4]{108}}{(-.057)^2 \times .002869}.$$

In this example observe that a minus sign is involved. Since we have already mentioned that there is no real logarithm of a negative number, we have then a seeming difficulty. However, under such circumstances we disregard the negative signs and proceed with the computation of the absolute values, and then prefix the appropriate sign to the result. Thus in this example we have

$$\frac{(-) \times (+)}{(-)^2 \times (+)} = \frac{(-)}{(+)} = -.$$

Proceeding, let $$x = \frac{.75 \times \sqrt[4]{108}}{(.057)^2 \times .002869}.$$

Then $\log x = (\log .75 + \frac{1}{4} \log 108) - (2 \log .057 + \log .002869)$. The work would be arranged as follows; copy and complete the example.

$\log .75 =$	$2 \log .057 =$	$\log x =$
$\frac{1}{4} \log 108 =$	$\log .002869 =$	$\therefore\ x = -$

Cologarithms. The division example $M \div N$ may be written as the product $M \times \left(\frac{1}{N}\right)$. From this we see that $\log \frac{M}{N} = \log M + \log \left(\frac{1}{N}\right)$. The $\log \left(\frac{1}{N}\right)$ is called the *cologarithm* of N and in abbreviated form is written colog N. In other words, the cologarithm of a number is the logarithm of the reciprocal of the number.

But $\log \left(\frac{1}{N}\right) = \log 1 - \log N = 0 - \log N$. In place of 0 we may write $10 - 10$. Why?

Hence, $\operatorname{colog} N = \log \left(\frac{1}{N}\right) = 10 - \log N - 10.$

Thus, to find the cologarithm of a number, subtract its logarithm from $10 - 10$.

For example, colog 3 = 10 − log 3 − 10 = 10 − 0.4771 − 10, which can be arranged:

$$\begin{array}{rrr} & 10 & -\ 10 \\ \log 3 = & 0.4771 & \\ \hline \operatorname{colog} 3 = & 9.5229 & -\ 10 \end{array}$$

$$\therefore\ \log \frac{M}{N} = \log M - \log N \quad \text{or} \quad \log M + \operatorname{colog} N.$$

The latter form, involving addition, is more convenient to use when the example contains a series of multiplications and divisions, as the following examples will show.

Illustrative examples.

Example 1. Find the logarithm of $\frac{3}{5}$, using cologarithms.

Solution

$$\begin{array}{rl} \log \frac{3}{5} = & \log 3 + \operatorname{colog} 5 \\ \log 3 = & .4771 \\ \operatorname{colog} 5 = & \underline{9.3010 - 10} \\ \therefore\ \log \frac{3}{5} = & 9.7781 - 10 \end{array} \qquad \left[\begin{array}{rrr} & 10 & -\ 10 \\ \log 5 = & .6990 & \\ \hline \operatorname{colog} 5 = & 9.3010 & -\ 10 \end{array}\right]$$

Example 2. Find the value of $\frac{39.6 \times 72.3}{3.61 \times .025}$.

Estimate

$$\frac{40 \times 72}{4 \times .025} = \frac{2880}{.1} = 28{,}800.$$

Solution

Let $x = \frac{39.6 \times 72.3}{3.61 \times .025}$.

$$\therefore\ \log x = \log 39.6 + \log 72.3 + \operatorname{colog} 3.61 + \operatorname{colog} .025.$$

$$\begin{array}{rrr} \log 39.6 = & 1.5977 & \\ \log 72.3 = & 1.8591 & \\ \operatorname{colog} 3.61 = & 9.4425 & -\ 10 \\ \operatorname{colog} .025 = & 1.6021 & \\ \hline \log x = & 14.5014 & -\ 10 \\ \log x = & 4.5014. & \\ \therefore\ x = & 31{,}720. & \end{array}$$

Assuming the numbers in Example 2 are measurements, since the least number of significant figures occurring is 2, the answer is 32,000.

Something to think about.

1. Is $\log \dfrac{1}{N} = \operatorname{colog} N$?

2. Is the logarithm of the reciprocal of a number always the cologarithm of that number?

3. Prove $\operatorname{colog} P^n = n \operatorname{colog} P$.

EXERCISES

Group A

Find the value of each of the following expressions:

1. $\dfrac{57.2 \times 3.09}{4.59 \times 21.7}$.

2. $\dfrac{.197 \times 32.87}{.655 \times .07275}$.

3. $\dfrac{72.48 \times .073}{8.92 \times (-21.45)}$.

4. $\dfrac{\sqrt[3]{1.44}}{.438}$.

5. $\dfrac{.849 \times 21.5}{\sqrt[3]{.0187}}$.

6. $\sqrt[3]{\dfrac{.9876 \times 63.35}{1.124}}$.

7. $\dfrac{(42.3)^3 \times .0135}{\sqrt[5]{136.7}}$.

8. $\dfrac{.0192(82.17)^2}{\sqrt[3]{.938}}$.

9. $\dfrac{(8.54)^3 \times \sqrt[5]{.148}}{.9532}$.

10. $\dfrac{(-.1872)^3 \times .3467}{(-75.8) \times .00381}$.

Group B

11. Given $\log 2 = .3010$, find the value of $\log 2^3$; $\log 2^{\frac{1}{2}}$; $\log \sqrt[3]{2}$; $\log 5$.

12. Given $\log 9 = .9542$, find the value of $\log 3$; $\log 27$; $\log \sqrt[3]{3}$; $\log 729$.

13. Given $\log 2 = .3010$ and $\log 3 = .4771$, find the value of $\log 6$; $\log \frac{3}{2}$; $\log \sqrt[3]{12}$; $\log 36$; $\log 1.5$.

14. Given $\log x = .5132$, find the value of $\log x^2$; $\log \sqrt{x}$; $\log \frac{1}{x}$; $\log .01\,x$.

15. Given $\log x^2 = 2.7226$, find $\log x$; $\log \sqrt{x}$; $\log \sqrt[3]{x^2}$; $\log \frac{1}{x}$; $\log x^4$.

16. Given $\log x = a$ and $\log y = b$, find the value in terms of a and b of $\log xy$; $\log xy^2$; $\log x^2y$; $\log x^2y^2$; $\log x\sqrt{y}$; $\log y\sqrt{x}$; $\log \frac{10}{xy}$.

17. If $v = at^2$, express $\log v$ in terms of the other letters.

18. Using logarithms, find the value of n in the formula $n = \frac{1}{2\,L}\sqrt{\frac{Mg}{m}}$ when $L = 78.5$, $M = 5468$, $g = 980$, $m = .0065$.

19. The formula $D = \sqrt[3]{\frac{W}{.5236\,(A - G)}}$ gives the diameter of a spherical balloon which is to lift a given weight W. By the use of logarithms find D if $A = .0807$, $G = .0056$, $W = 1250$.

20. Find the value of r in the formula $r = \sqrt[3]{\frac{3\,V}{4\,\pi}}$ when $\pi = 3.14$ and $V = 56.3$.

Group C

21. The formula for the volume of a circular cylinder is $V = \pi r^2 h$. Find by logarithms the value of r when $V = 912$, $h = 13.8$, and $\pi = 3.14$.

22. The formula $t = \sqrt{\frac{\pi l}{g}}$ relates to a pendulum. Find the value of t when $\pi = 3.14$, $l = 35.8$, and $g = 980$.

23. Write $\log_7 35 = x$ as an exponential equation. From this equation find the value of $\log_7 35$.

24. Find the value of $\log_3 17$.

25. At a point 182.7 feet from the foot of a building on level ground the angle of elevation of the top is 52°. Find, by logarithms, the height of the building.

26. Compute the value of the expression $\sqrt[3]{.3468 \cos 113°}$.

27. Explain why the law of cosines does not lend itself readily to treatment by logarithms.

A business application of logarithms. One of the most important computations in the business world is the finding of compound interest. This would be very tedious were it not that the use of logarithms simplifies the computation.

Let us first derive a formula for compound interest.

Suppose that a sum of money P (the principal) is put into a bank at the rate of 6% interest compounded annually. Observe how the following table helps in building up a formula for the amount that P dollars will become when invested at 6%, interest compounded annually.

At end of year	Principal	Will earn interest	And the amount will be
1	P	$.06\ P$	$P + .06\ P = P(1.06)$
2	$P(1.06)$	$.06\ P(1.06)$	$P(1.06) + .06\ P(1.06)$ $= P(1.06)(1 + .06) = P(1.06)^2$
3	$P(1.06)^2$	$.06\ P(1.06)^2$	$P(1.06)^2 + .06\ P(1.06)^2$ $= P(1.06)^2(1 + .06) = P(1.06)^3$

Generalizing the foregoing data, if the principal is P, the rate of interest r, the number of years n, and the amount A, we have

$$A = P(1 + r)^n.$$

If the interest is compounded semi-annually or quarterly, the same formula can be used provided we understand n to be the number of interest periods and r the rate for one interest period.

Illustrative examples.

Example 1. If \$3500 is invested at 4%, interest compounded annually, for a period of 10 years, find the amount.

Solution $A = P(1 + r)^n.$

In this example, $P = 3500$, $r = .04$, and $n = 10$.

$$\begin{aligned}
A &= 3500(1 + .04)^{10}.\\
\log A &= \log 3500 + 10 \log 1.04.\\
\log 3500 &= 3.5441\\
10 \log 1.04 &= \underline{.1700}\\
\log A &= 3.7141\\
A &= \$5178.
\end{aligned}$$

Example 2. What sum of money placed at 5%, interest compounded semi-annually, will amount to \$5000 at the end of 20 years?

Solution

There are 40 interest periods in 20 years, and the rate for one period (half year) is $2\frac{1}{2}$%. Hence, in the example,

$$\begin{aligned}
A &= 5000,\ r = .025, \text{ and } n = 40.\\
A &= P(1 + r)^n.\\
5000 &= P(1 + .025)^{40}.\\
\log 5000 &= \log P + 40 \log 1.025.\\
\therefore \log P &= \log 5000 - 40 \log 1.025.\\
\log 5000 &= 3.6990\\
40 \log 1.025 &= \underline{.4280}\\
\log P &= 3.2710\\
\therefore P &= \$1867.
\end{aligned}$$

Something to think about. In 1623 the Indians sold Manhattan Island for \$24. Now the value of the island is about \$10,000,000,000. If the purchase price had been invested in 1623 at 6% interest compounded annually, would the amount this year be more or less than the \$10,000,000,000?

EXERCISES

If interest is compounded annually, find the amount of:

1. \$500 for 6 years at 6%.
2. \$275 for 3 years at 4%.
3. \$795 for 5 years at 3%.
4. \$2170 for 4 years at $5\frac{1}{2}$%.
5. \$3700 for 6 years at $3\frac{1}{2}$%.
6. \$1950 for 15 years at $4\frac{1}{2}$%.

If interest is compounded semi-annually, find the amount of:

7. \$325 for 5 years at 6%.
8. \$728 for 4 years at 5%.
9. \$856 for 3 years at 4%.
10. \$3500 for 10 years at $4\frac{1}{2}$%.
11. \$2700 for 8 years at $5\frac{1}{2}$%.
12. \$1270 for 5 years at $3\frac{1}{2}$%.

If interest is compounded quarterly, find the amount of:

13. \$375 for 4 years at 6%
14. \$525 for 5 years at 5%.
15. \$850 for 3 years at 4%.
16. \$7000 for 10 years at $4\frac{1}{2}$%.
17. \$3700 for 8 years at $5\frac{1}{2}$%.
18. \$2520 for 5 years at $3\frac{1}{2}$%.

Find the principal which will amount to:

19. \$8500 at the end of 5 years at 4%, interest compounded annually.

20. \$7500 at the end of 10 years at 5%, interest compounded annually.

21. \$10,000 at the end of 20 years at 6%, interest compounded annually.

22. \$5000 at the end of 10 years at 3%, interest compounded semi-annually.

23. \$6000 at the end of 15 years at 5%, interest compounded quarterly.

Exponential Equations

Problem. If \$5000 were deposited in a bank at 5%, interest compounded annually, in how many years would it amount to \$10,000?

Substituting in the formula $A = P(1 + r)^n$,

$$10{,}000 = 5000(1 + .05)^n$$

where n is to be found. Here you observe a type of equation with the unknown as an exponent, the value of which cannot be discovered by inspection. This type of equation, as you know, is called an exponential equation. We shall now learn how exponential equations are solved by using logarithms.

Illustrative example. Solve $12^x = 45$.

Solution

Taking logarithms of both members, we have

$$x \log 12 = \log 45.$$

$$\therefore x = \frac{\log 45}{\log 12}.$$

$\dfrac{\log 45}{\log 12}$ is the quotient of two logarithms and not the logarithm of a quotient. Do not confuse the expressions $\log \dfrac{x}{y}$ and $\dfrac{\log x}{\log y}$. In the latter case we must find the logarithms of the numerator and denominator, and then actually divide.

Thus $$x = \frac{1.6532}{1.0792} = 1.532.$$

Now to return to our problem.

Illustrative example. If \$5000 were deposited in a bank at 5%, interest compounded annually, in how many years would it amount to \$10,000?

Solution

It is necessary to solve the equation

$$10{,}000 = 5000(1 + .05)^n.$$

Taking logarithms of both members, we have

$$\log 10{,}000 = \log 5000 + n \log 1.05.$$

Solving for n, $$n = \frac{\log 10{,}000 - \log 5000}{\log 1.05}.$$

$$n = \frac{4.0000 - 3.6990}{.0212}$$

$$= \frac{.3010}{.0212} = 14.2.$$

Hence, mathematically the answer is approximately 14.2 years, but actually, since interest is compounded annually, the full amount could not be realized until the end of the 15th year.

EXERCISES

Solve:

1. $2^x = 25$.	**6.** $2.5^x = 17$.	**11.** $1.05^x = 3.6$.
2. $3^x = 50$.	**7.** $1.8^x = 35$.	**12.** $1.06^x = 5.2$.
3. $4^x = 28$.	**8.** $1.3^x = 29$.	**13.** $2^{x+1} = 42$.
4. $5^x = 30$.	**9.** $1.03^x = 6$.	**14.** $3^{x-1} = 70$.
5. $10^x = 75$.	**10.** $1.04^x = 3$.	**15.** $5^{2x+1} = 175$.

16. In how many years will \$3000 amount to \$10,000 if invested at 4%, interest compounded annually?

17. In how many years will \$2170 amount to \$3150 if invested at 5%, interest compounded annually?

18. In what length of time would a sum of money at 5% double itself, if the interest were compounded annually?

19. How long would it take a sum of money to treble itself at 4%, interest compounded annually?

20. In how many years will \$1 double itself at 6%, interest compounded semi-annually?

The Slide Rule

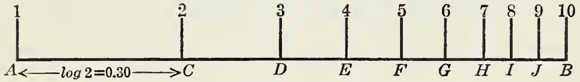
Engineers, mechanics, statisticians, chemists, architects, accountants, business men, and surveyors, whose work calls for a large amount of arithmetical computation, are always on the watch for short cuts and labor-saving devices for making approximate calculations rapidly.

The slide rule is such a device. It is based on the principles of logarithms, but provides a shorter method of calculating than is possible with logarithmic tables. In it the table of mantissas is replaced by a "logarithmic scale." Let us construct such a logarithmic scale.

Construction of a logarithmic scale. Let us make a table of the logarithms of the integers from 1 to 10:

Number	1	2	3	4	5	6	7	8	9	10
Logarithm	0.00	0.30	0.48	0.60	0.70	0.78	0.85	0.90	0.95	1.00

On a strip of paper lay off a line AB representing a unit of length. A convenient length for AB would be ten centimeters.



Then $AB = 1 = \log 10$. Then lay off the logarithms in the table and label each with its corresponding number.

Thus $\quad AC = .30$ of the unit $= \log 2$,

$\qquad AD = .48$ of the unit $= \log 3$, etc.

Therefore, the measure of the distance from 1 to any number is the logarithm of that number. AB is called the *logarithmic scale.* Observe that we have really pictured on this scale the mantissas of the logarithms of the integers from 1 to 10, or from 10 to 100, or from 100 to 1000, etc., because the mantissas of, say, 9, 90, 900, etc., are the same.

These divisions are called the *main divisions.*

In the same way each of the main divisions can be subdivided into tenths, called *secondary divisions.* Again, each of these secondary divisions may be further subdivided into tenths. These last divisions enable us to represent the logarithms of numbers of three figures. Below is a section of a logarithmic scale so divided. Find on this scale log 165; log 234; log 316.

By making AB sufficiently long the number of graduations or divisions may be increased. The most common scale, which is 10 inches long, gives the logarithms of numbers of three figures only because further subdivisions would make the intervals too small for practical use.

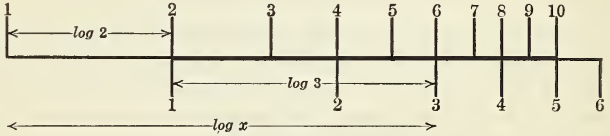
Performing operations with the logarithmic scale. The principles that you have learned in connection with actual logarithms, apply to the use of the logarithmic scale. The only difference is that the logarithms on the scale are represented by lengths instead of numbers as in the table.

Illustrative example. Multiply 2 by 3 on the logarithmic scale.

Solution

Let $x = 2 \times 3$.

Since $\log x = \log (2 \times 3) = \log 2 + \log 3$, the required answer is obtained from the addition of log 2 and log 3 on the scale.



Thus we lay off on the scale the length from 1 to 2 (log 2). We then measure the length of log 3 with our dividers and add it to the length from 1 to 2 (log 2) just as we would lay off 2 inches and 3 inches on a foot rule. Then the addition of the two distances represents a distance $\log x$, which we observe is log 6. Therefore $x = 6$. The other operations done with the aid of logarithms can also be performed on the logarithmic scale.

The slide rule. The slide rule, which is used most widely, has four logarithmic scales usually referred to as A, B, C, D. Scales C and D are alike and resemble the one used above, while A and

B are alike but have smaller divisions. In order to fix points accurately in performing operations, a runner with a hair line is attached and can be moved along the scale.

A complete and detailed explanation of slide rule operations cannot be given in a book of this type. No one can learn to use a slide rule without having a slide rule to work with. Inexpensive slide rules are now manufactured at a cost within easy reach of the high school student. A manual of instructions is furnished with each rule, and the best results are obtained with a slide rule in your hands, a manual of instructions in front of you, and a teacher who knows how to guide you.

Extended Use of Tables

Problem. Solve the equation $8 \sin^2 x - 6 \sin x + 1 = 0$ for all values of x, between 0° and 360°. Determine the approximate values of x correct to the nearest minute.

First it will be necessary to become familiar with a more elaborate table of natural functions. Up to this point you have used the table on page 3. But, since in the above example it is required to work with degrees and minutes, Table II on pages 574 to 578 should be used. You will observe that this new table, unlike the old, gives the values of the sine, cosine, tangent, and cotangent only. To find the secant and cosecant you may use the reciprocal formulas, $\sec A = \dfrac{1}{\cos A}$ and $\csc A = \dfrac{1}{\sin A}$.

The table of natural functions. Table II gives the values of the sine, cosine, tangent, and cotangent of angles from 0° to 90°, at ten minute intervals, correct to four decimal places. For angles from 0° to 45°, the degrees and minutes are read *downward* in the left-hand column. The sine, cosine, tangent, and cotangent values appear to the right of the angles in the order named, the name of the function being at the top of the column. For angles from 45° to 90° the degrees and minutes are read *upward* in the right-hand column. The tangent, cotangent, sine, and cosine values appear to the left of the angles in the order named, the name of the function being at the bottom of the column.

Since the new table gives the values of the functions for each ten minutes, we can calculate by interpolation the values of the functions when the angle is expressed to the nearest minute. Conversely, from a given value of a function, we can determine the angle correct to the nearest minute. It is important to remember that as the angle increases, the sine and tangent increase but the cosine and cotangent decrease.

Illustrative examples.

Example 1. Find the value of sin 25° 20′.

Solution

Turn the pages of the table until you find 25°. Look down the column from 25° until you come to 20′. In the sine column on the line with 20′, you will find .4279, the required answer.

$$\therefore \sin 25° \, 20' = .4279.$$

Example 2. Find the value of cos 80° 50′.

Solution

Since we have an angle greater than 45°, turn to the page upon which 80° appears in the right-hand column. Look up the column from 80° until you come to 50′. In the cosine column (read from the bottom), on the line with 50′, you will find .1593, the required answer.

Thus $$\cos 80° \, 50' = .1593.$$

Example 3. Find the value of tan 75° 43′.

Solution

The value of this function lies between the values in the table for tan 75° 40′ and tan 75° 50′. By interpolating, as shown on the opposite page, we find the required value.

$$\begin{array}{lllll}
 & \textit{Angle} & & \textit{Tangent} & \\
 & 75^\circ\, 40' & \longrightarrow & 3.9136 & \\
 & & \text{Increase } 3' & & \\
\text{Increase } 10' & 75^\circ\, 43' & & ? & \text{Increase } .0481 \\
 & 75^\circ\, 50' & \longrightarrow & 3.9617 &
\end{array}$$

$$\tfrac{3}{10} \times .0481 = .0144.$$

$$\therefore \tan 75^\circ\, 43' = 3.9136 + .0144 = 3.9280.$$

Example 4. Find, correct to the nearest minute, the angle whose cosine is .2914; *i.e.*, if

$$\cos x = .2914,\ x = ?$$

Solution

By turning to the cosine columns of the table, you will find that .2914 lies between .2924 or cos 73° 0′ and .2896 or cos 73° 10′. By interpolating, we find the required angle. Remember that the cosine decreases as the angle increases from 0° to 90°.

$$\begin{array}{lllll}
 & \textit{Angle} & & \textit{Cosine} & \\
 & 73^\circ\, 00' & \longleftarrow & .2924 & \\
\text{Increase } 10' & ? & \text{Decrease } .0010 & .2914 & \text{Decrease } .0028 \\
 & 73^\circ\, 10' & \longleftarrow & .2896 &
\end{array}$$

$$\frac{.0010}{.0028} \text{ of } 10' = 4'.$$

$$\therefore x = 73^\circ + 4' = 73^\circ\, 4'.$$

Now to return to the problem on page 453.

Example 5. Solve the equation $8 \sin^2 x - 6 \sin x + 1 = 0$ for all values of x between 0° and 360°. Determine the approximate values of x correct to the nearest minute.

Solution

$$8 \sin^2 x - 6 \sin x + 1 = 0.$$
$$(2 \sin x - 1)(4 \sin x - 1) = 0.$$
$$2 \sin x - 1 = 0.$$

$$\sin x = \tfrac{1}{2} = .5000$$
$$x_1 = 30^\circ.$$
$$x_2 = 180^\circ - 30^\circ = 150^\circ.$$
$$4 \sin x - 1 = 0.$$
$$\sin x = \tfrac{1}{4} = .2500.$$

	Angle		*Sine*	
	$14^\circ\, 20'$	$\longleftarrow$	.2476	
Increase $10'$	?	Increase .0024	.2500	Increase .0028
	$14^\circ\, 30'$	$\longleftarrow$	.2504	

$$\frac{.0024}{.0028} \text{ of } 10' = \tfrac{6}{7} \times 10' = 9'.$$
$$\therefore x_3 = 14^\circ\, 29'.$$
$$x_4 = 180^\circ - 14^\circ\, 29' = 165^\circ\, 31'.$$
$$\therefore x = 30^\circ,\ 150^\circ,\ 14^\circ\, 29',\ 165^\circ\, 31'.$$

To check any value of x, substitute that value in the original equation.

EXERCISES

Use Table II throughout this exercise.

Find the value of each of the following:

1. sin 26° 30′.
2. cos 88° 10′.
3. tan 44° 50′.
4. cot 56° 40′.
5. sin 10° 20′.
6. cos 44° 10′.
7. cot 3° 50′.
8. tan 80° 20′.
9. sin 12° 13′.
10. cos 56° 56′.
11. tan 85° 11′.
12. cot 38° 17′.
13. sin 89° 1′.
14. cos 45° 18′.
15. tan 44° 45′.

Find the angle in degrees and minutes whose:

16. sine is .0291.
17. cosine is .9994.
18. tangent is .0058.
19. cotangent is 31.242.
20. sine is .9744.
21. sine is .4772.

In each of the following find the angle correct to the nearest minute:

22. $\sin x = .7090.$

23. $\tan x = 1.0751.$

24. $\cos x = .7614.$

25. $\cot x = .8200.$

26. $\sin y = .5903.$

27. $\cos z = .8464.$

28. $\tan A = 1.5640.$

29. $\cot B = .6149.$

30. $\sin x = .5296.$

31. $\cos C = .5131.$

Solve the following equations for all values of x between 0° and 360°. Determine the approximate values of x correct to the nearest minute.

32. $6 \cos^2 x - 5 \cos x + 1 = 0.$

33. $3 \sin^2 x = 5 \sin x - 2.$

34. $\tan^2 x - 3 \tan x + 2 = 0.$

35. $3 \cot^2 x - 5 \cot x + 2 = 0.$

36. $4 \tan^2 x + \tan x = 5.$

37. $5 \sin^2 x = 4 - \sin x.$

38. $\sin x(10 \cos x - 9) = 0.$

39. $\cos x(7 \sin x - 5) = 0.$

40. $\tan x(10 \tan x + 13) = 0.$

Logarithms of the trigonometric functions. You have already seen how logarithms reduced the labor and time involved in numerical calculations. You will also remember the tedious multiplications and divisions that had to be performed in solving some of the trigonometric problems in Chapter I. Logarithms can be used to shorten and simplify computations involving trigonometric functions, but first it will be necessary to learn about a new table giving the logarithms of such functions.

Table III on pages 579–583 shows the logarithms of the trigonometric functions. It is really two tables in one.

To write the logarithm of sin 64°, we could first look up the value of sin 64° in the table of natural functions (Table II), where we would find sin 64° = .8988. Then we could write the logarithm of .8988, using Table I. The work may be written as shown at the top of page 458.

$$\sin 64° = .8988.$$
$$\log .8988 = 9.9537 - 10.$$
$$\therefore \log \sin 64° = 9.9537 - 10.$$

Table III contains the results obtained by combining these two operations. Therefore, it gives the logarithms of the trigonometric functions in *one reading.*

As in Table II, the logarithms of the sine, cosine, tangent, and cotangent only are given. The logarithms of the secant and cosecant can be found by using the reciprocal relations

$$\sec A = \frac{1}{\cos A} \quad \text{and} \quad \csc A = \frac{1}{\sin A}.$$

Or in logarithmic form,

$$\log \sec A = \text{colog} \cos A \quad \text{and} \quad \log \csc A = \text{colog} \sin A.$$

Description and use of Table III. The sines and cosines of positive angles less than 90° are less than 1. Also the tangents of positive angles less than 45° and the cotangents of angles greater than 45°, up to and including 90°, are less than 1. Consequently the logarithms of these functions have negative characteristics. To avoid them, the characteristic of each of these logarithms is printed 10 too large throughout the table. Therefore, you must remember to add -10 after each of these logarithms.

In this table log sin x, log cos x, log tan x, and log cot x are given for every ten minutes from 0° to 90°. The table can be used to find the logarithmic functions when the angle is given or to find the angle when a logarithmic function is given.

If the angle is less than 45°, the degrees and minutes are read downward in the left-hand column. The names L Sin (Log Sin), L Cos, L Tan, and L Cot appear at the top of the columns. If the angle is greater than 45°, the degrees and minutes are read upward in the right-hand column, and the names L Tan, L Cot, L Sin, and L Cos are at the bottom of the columns.

Since the table gives the values of the logarithms of the functions for every 10 minutes, we can calculate by interpolation these logarithms for angles expressed to the nearest minute. Conversely, from the logarithm of a function, we can determine the angle correct to the nearest minute.

Illustrative examples.

Example 1. Find log cos 49° 16′.

Solution

In the table, this logarithm lies between log cos 49° 10′ and log cos 49° 20′. Consequently we interpolate as follows:

$$\begin{array}{lllll}
 & \textit{Angle} & & \textit{Log cos} & \\
 & 49^\circ\, 10' & \longrightarrow & 9.8155-10 & \\
 & & \text{Increase } 6' & & \\
\text{Increase } 10' & 49^\circ\, 16' & & ? & \text{Decrease } .0015 \\
 & 49^\circ\, 20' & \longrightarrow & 9.8140-10 &
\end{array}$$

$$\tfrac{6}{10} \times .0015 = .0009.$$

$$\therefore \log \cos 49^\circ\, 16' = (9.8155 - 10) - .0009$$
$$= 9.8146 - 10.$$

Example 2. Find correct to the nearest minute the angle whose log tan is 9.7273 − 10.

Solution

9.7273 − 10 lies between 9.7257 − 10 and 9.7287 − 10.

$$\begin{array}{llllll}
 & \textit{Angle} & & \textit{Log tan} & \\
 & 28^\circ\, 00' & \longleftarrow & 9.7257-10 & \\
\text{Increase } 10' & ? & \text{Increase } .0016 & 9.7273-10 & \text{Increase } .0030 \\
 & 28^\circ\, 10' & \longleftarrow & 9.7287-10 &
\end{array}$$

$$\frac{.0016}{.0030} \times 10' = 5'.$$

$$\therefore x = 28^\circ\, 00' + 5' = 28^\circ\, 5'.$$

EXERCISES

Find the value of each of the following, using Table III:

1. log sin $38° 20'$.
2. log cos $85° 40'$.
3. log tan $12° 36'$.
4. log cot $7° 17'$.
5. log tan $46° 57'$.
6. log cot $75° 47'$.
7. log cos $44° 28'$.
8. log sin $4° 6'$.
9. log cot $50° 52'$.
10. log cos $73° 41'$.
11. log tan $11° 23'$.
12. log sin $40° 4'$.
13. log cos $12° 18'$.
14. log cot $19° 18'$.

Find the value of x in each of the following:

15. $\log \cos x = 8.2419 - 10$.
16. $\log \sin x = 9.5859 - 10$.
17. $\log \cos x = 9.9992 - 10$.
18. $\log \sin x = 9.7744 - 10$.
19. $\log \tan x = 10.0608 - 10$.
20. $\log \cot x = 11.0034 - 10$.
21. $\log \cot x = 9.3397 - 10$.
22. $\log \tan x = 1.2806$.

Find the following angles correct to the nearest minute:

23. $\log \sin x = 8.8620 - 10$.
24. $\log \cos x = 9.9960 - 10$.
25. $\log \tan x = 11.1737 - 10$.
26. $\log \cot x = 9.1905 - 10$.
27. $\log \cos x = 9.5462 - 10$.
28. $\log \sin B = 9.7135 - 10$.
29. $\log \tan C = 9.9999 - 10$.
30. $\log \sin A = 9.8868 - 10$.
31. $\log \tan B = 8.9770 - 10$.
32. $\log \cot A = 11.0000 - 10$.

Find:

33. colog tan $18° 27'$.
34. colog cos $81° 73'$.
35. colog sin $5° 23'$.
36. log sec $17° 20'$.
37. log csc $73° 48'$.
38. colog cot $65° 17'$.
39. colog sin $44° 44'$.
40. colog cos $45° 46'$.
41. colog sec $45° 9'$.
42. colog csc $83° 38'$.

Who Invented Logarithms?

The most practical device for the purposes of modern calculation is logarithms, invented early in the 17th century by John Napier, a Scotch nobleman (1550–1617). It is rather curious that whereas we now regard our knowledge of exponents as the basis of logarithms, yet Napier formulated tables of logarithms before exponents came into definite use. That the theory of logarithms was a natural sequence from the exponential equation was not understood until 150 years after Napier, when Euler gave a systematic exposition of the relation between exponents and logarithms.

Napier did not use the base 10 which is used generally today. Henry Briggs, a professor of geometry in London, after discussing the possibilities with Napier, suggested improvements such as the base 10, so that the characteristic could be determined by inspection. Briggs eventually published his tables with the base 10 and used the word *mantissa* to mean the fractional part of the logarithm.

CUMULATIVE REVIEW

Chapters XI, XII, and XIII

1. Which of these statements are true? Which are false?

(*a*) $\sin(x + y) + \sin(x - y) = 2\sin x$.

(*b*) $\tan(x - 180°) = \tan x$. (*e*) $-3a^0 = -3$.

(*c*) $32a^{\frac{4}{5}} = \sqrt[5]{32a^4}$. (*f*) $8 \times 256 = 2^{11}$.

(*d*) $-.25^{\frac{3}{2}} = -.125$. (*g*) $\log_6 36 = \log_{10} 100$.

(*h*) The characteristic of 185.782 is the same as the characteristic of 782.185.

(*i*) $\dfrac{\log b}{\log a} = \log b - \log a$.

(*j*) If $\log 3 = .4771$ and $\log 2 = .3010$, then $\log 5 = .7781$.

2. Complete each of the following statements:

(*a*) Expressed as a product, $\cos 6x + \cos 4x = ?$

(*b*) The two smallest positive angles which satisfy the equation $\sin 4x = 0$ are ? degrees and ? degrees.

(*c*) When $a = 2$, the value of $a^7 \cdot a^{-3} \cdot a^{-5}$ is ? .

(*d*) When $a = 3$ and $b = 2$, the value of $\dfrac{1}{a^{-1} + b^{-1}}$ is ? .

(*e*) The number of significant zeros in 0.0708 is ? .

(*f*) If $\log x = 2.5478$, then $x = ?$

(*g*) If $\log 7 = .8451$ and $\log 4 = .6021$, then $\log 28 = ?$

(*h*) $\log 38.47 = ?$

(*i*) $\log \sin 59° 17' = ?$

(*j*) If $\log x = 6.6315$, then $\sqrt[3]{x} = ?$

3. Solve for y and check: $\tan^{-1} 2y + \tan^{-1} 3y = \dfrac{3\pi}{4}$.

4. If $A = \sin^{-1} \frac{1}{2}\sqrt{2}$,

(*a*) state in which quadrants angle A may lie.

(*b*) give the principal value of angle A in degrees and also in radians.

(*c*) write the other value, less than 360°, which angle A may have.

(*d*) construct the principal value of angle A.

(*e*) give the value of $\cos (\sin^{-1} \frac{1}{2}\sqrt{2})$.

5. Find the value of: $\dfrac{1}{x^{\frac{3}{2}}} - 7x^0 + 1^{\frac{4}{5}} - \left(\dfrac{1}{x}\right)^{\frac{1}{2}} + \dfrac{1}{9}x^{1\frac{1}{2}}$ when $x = 9$.

6. (*a*) Divide: $e^{-2x} + 2 + e^{2x}$ by $e^{-x} + e^x$.

(*b*) What is the numerical value of the quotient when $x = 0$?

7. Expand $(x^{-1} - y^{\frac{1}{4}})^2$.

8. Find the value of $(1.025)^{22}$.

9. Find the value of $\sqrt[5]{.002136}$.

10. Find the value of $\dfrac{(1.52)^4 \times .00312}{\sqrt[3]{12.3}}$.

11. If x is an obtuse angle such that $\sin x = \dfrac{(9.635)\sqrt{4.086}}{(12.74)^2}$, compute by logarithms the value of x correct to the nearest minute.

12. Mr. Smith deposits \$5950 in a savings bank that pays at the rate of 4% interest, compounded semi-annually. What amount, to the nearest dollar, will Mr. Smith have in the bank at the end of 8 years?

CHAPTER XIV. THE SOLUTION OF TRIANGLES

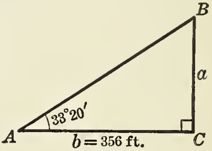
To measure is to know. — Kepler.

Solution of Right Triangles by Logarithms

Much of the computation involved in your earlier work in solving right triangles can be simplified and shortened by the use of logarithms.

Illustrative examples.

Example 1. A tower standing on a level plain subtends an angle of 33° 20′ at a point on the plain 356 feet from the foot of the tower. Find the height of the tower.

Given	*To find*	
$C = 90°.$ $A = 33° 20'.$ $b = 356$ ft.	$a = ?$ ft.	

Solution

$$
\begin{aligned}
\frac{a}{b} &= \tan A.\\
\therefore\ a &= b \tan A.\\
\log a &= \log b + \log \tan A.\\
\log b &= 2.5514\\
\log \tan A &= \underline{9.8180 - 10}\\
\log a &= 12.3694 - 10\\
&= 2.3694.\\
\therefore\ a &= 234.1 \text{ ft.}
\end{aligned}
$$

$$
\begin{array}{cccc}
 & \textit{Number} & & \textit{Log} \\
 & 2340 & \longleftarrow & .3692 \\
 & & .0002 & \\
10 & ? & & .3694 \quad .0019 \\
 & 2350 & \longleftarrow & .3711
\end{array}
$$

$$\frac{.0002}{.0019} \times 10 = \frac{2}{19} \times 10 = 1.$$

$\therefore$ number is 2341.

Check

$$\cot A = \frac{b}{a}. \qquad (234.1)(1.5204) = 356.$$

$$a \cot A = b. \qquad 356 = 356 \text{ approx.}$$

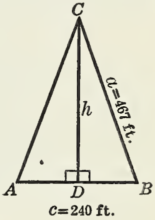
Since the measurement of side b is given correct to 3 significant figures, we shall state the answer correct to the same degree of accuracy.

$\therefore$ the tower is 234 feet high.

Since the measurement of a should be stated correct to 3 significant figures, the value could have been found directly by inspection. From the table you can see that the mantissa .3694 is much nearer to the mantissa of 234 than to the mantissa of 235. Hence, the answer is $a = 234$ ft.

Example 2.

Given	*To find*
Isosceles $\triangle ABC$ with $AC = BC$. $a = 467$ ft. $c = 240$ ft.	$B = ?$ $h = ?$ ft.



Solution

$$\cos B = \frac{DB}{CB}.$$

$$\cos B = \tfrac{120}{467}.$$

$$\log \cos B = \log 120 + \operatorname{colog} 467.$$

$$\begin{aligned} \log 120 &= 2.0792. \\ \operatorname{colog} 467 &= \underline{7.3307 - 10} \\ \log \cos B &= 9.4099 - 10. \\ \therefore B &= 75^\circ 7'. \end{aligned}$$

$$h^2 + (DB)^2 = a^2.$$

$$\begin{aligned} h^2 &= a^2 - (DB)^2 \\ &= (a + DB)(a - DB) \\ &= (467 + 120)(467 - 120) \\ &= (587)(347). \end{aligned}$$

$$\begin{aligned} 2 \log h &= \log 587 + \log 347. \\ \log 587 &= 2.7686 \\ \log 347 &= \underline{2.5403} \\ 2 \log h &= 5.3089 \\ \log h &= 2.6545. \\ \therefore h &= 451.3. \end{aligned}$$

Check

$$\frac{h}{a} = \sin B. \qquad 451.3 = (467)(.9665)$$

$$\therefore\ h = a \sin B. \qquad 451.3 = 451.3 \text{ approx.}$$

As the measurements of the sides are given correct to 3 significant figures, we shall state the answers correct to the same degree of accuracy.

$$\therefore\ h = 451 \text{ ft. and } B = 75°\,7'.$$

EXERCISES ON THE RIGHT TRIANGLE

Exercises 1–25 refer to a right triangle ABC, in which angle $C = 90°$. Write the formula expressing each of the required parts in terms of the given parts:

1. Given: A, c; required: a, b, B.

2. Given: A, a; required: B, b, c.

3. Given: A, b; required: B, a, c.

4. Given: a, c; required: A, B, b.

5. Given: a, b; required: A, B, c.

Using logarithms, solve the following right triangles, given:

6. $c = 476$, $A = 14°\,45'$.

7. $c = 521$, $B = 49°\,45'$.

8. $a = 1.84$, $b = 3.17$.

9. $a = 843$, $B = 55°\,45'$.

10. $b = 56.4$, $c = 70.8$.

11. $b = 12.5$, $B = 57°\,46'$.

12. $c = 4590$, $B = 60°\,50'$.

13. $c = 16.4$, $a = 5.87$.

14. $b = 13.1$, $A = 66°\,12'$.

15. $b = 50.8$, $c = 60.9$.

16. $c = 377.3$, $B = 52°\,40'$.

17. $c = 9641$, $a = 5832$.

18. $a = 10$, $B = 50°$.

19. $c = 250$, $b = 95.67$.

20. $a = 8000$, $b = 8.943$.

21. $b = 1765$, $A = 22°\,20'$.

22. $c = 138.7$, $A = 41°\,26'$.

23. $b = 841$, $a = 3366$.

24. $a = 136.2$, $b = 302.4$.

25. $c = 185.6$, $B = 55°\,44'$.

26. A flagpole 80 feet high casts a shadow 50.7 feet long on level ground. What is the angle of elevation of the sun?

27. A tower casts a shadow which is three-fifths its own length. What is the angle of elevation of the sun?

28. A hill rises 425 feet in a distance of 4025 feet measured up the hillside. Find the angle at which the hill slopes.

29. The shadow of a tree is 40.84 feet on level ground when the elevation of the sun is known to be $58° 44'$. Find the height of the tree.

30. The top of a tree, broken by the wind, strikes the ground 32 feet from the part of the tree left standing, and makes an angle of $35° 54'$ with the ground. Find the original height of the tree.

31. Two poles 25.1 feet and 35.7 feet are set vertically 39.4 feet apart. What is the length of a wire connecting their tops?

32. From the top of a cliff 685 feet above sea level, the angles of depression of two boats, in line with the observer, are $15° 15'$ and $25° 42'$ respectively. What is the distance between the boats?

33. A building surmounted by a flagpole 20 feet high stands on level ground. From a point on the ground the angles of elevation of the top and bottom of the pole are $53° 10'$ and $45° 20'$ respectively. How high is the building?

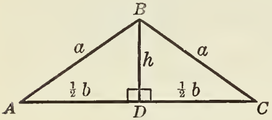
34. A lighthouse whose top is 114 feet above the level of the water and two buoys are situated in the same straight line. From the top of the lighthouse the angles of depression of the two buoys are $37°$ and $15°$ respectively. What is the distance between the two buoys?

35. A ladder 50 feet long rests against a building and makes an angle of $30°$ with it. If the foot of the ladder were moved 10 feet farther from the wall, how far down the wall would the top of the ladder fall?

EXERCISES ON THE ISOSCELES TRIANGLE

In the isosceles triangle ABC, $\angle B$ is the vertex angle and h is the altitude to the base.

Write the formula expressing each of the parts not given in terms of the given parts. Given:



1. A, b.	**4.** B, a.	**7.** a, h.
2. A, a.	**5.** B, b.	**8.** a, b.
3. A, h.	**6.** B, h.	**9.** h, b.

Using logarithms, solve the following isosceles triangles given:

10. $B = 85° 4'$, $a = 275$.

11. $b = 65$, $a = 90$.

12. $A = 48°$, $b = 24$.

13. $a = 241$, $A = 42°$.

14. $b = 40$, $h = 30$.

15. $B = 68° 34'$, $b = 437.2$.

16. In an isosceles triangle the altitude to the base is equal to the base. Find the three angles of the triangle. Is this triangle determined; that is, do you know the size of each angle and the length of each side?

17. The legs of an isosceles triangle are each 15 inches and the altitude to the base is equal to the base. Find the vertex angle and the length of the altitude.

18. If a chord of a circle is one-half of the radius, what is the size of the angle subtended by the chord at the center?

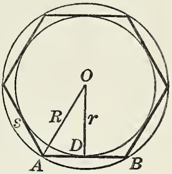
19. The angle made by the roof of a barn with the horizontal is 45°. If the width of the barn is 36 feet, what is the length of the rafters?

20. A searchlight situated on a coast has a range of 40 miles. A ship sails on a line parallel to the coast and 12 miles from it. What is the greatest distance covered by the ship if it is to remain within range of the light? What angle does this distance subtend at the light?

EXERCISES ON THE REGULAR POLYGON

Examples 1 to 14 refer to regular polygons in which:

n = number of sides.
s = length of one side.
R = radius of circumscribed circle.
r = radius of inscribed circle.
p = perimeter.



Write the formula expressing each of the required parts in terms of the given parts:

	Given	Required
1.	R, s	$r, \angle AOD$
2.	R, r	$s, \angle AOD$
3.	n, s	$p, \angle AOB$
4.	R, n	$\angle AOD, r$
5.	r, n	$\angle AOD, R, s$
6.	r, s	$\angle AOD, R$
7.	$\angle AOD, R$	$n, s, p.$

Using logarithms, solve the following examples based on regular polygons:

	Given	Required
8.	$n = 6,\ s = 12.86$	$\angle AOB, R, r$
9.	$n = 12,\ R = 16.31$	$\angle AOD, r, s$
10.	$n = 15,\ r = 8.502$	R, s, p
11.	$n = 9,\ r = 10.47$	R, p
12.	$R = 3.753,\ s = 3.753$	n, r
13.	$n = 17,\ R = 31$	r, s
14.	$n = 11,\ s = 15.7$	R, r

15. The length of a side of a regular octagon is 3.75 inches. Find the radius of the inscribed circle.

16. Find the perimeter of a regular decagon inscribed in a circle whose radius is 12.53 inches.

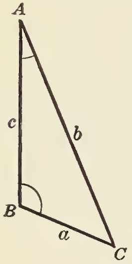
17. The radius of a circle inscribed in an equilateral triangle is 9.7 inches. Find the length of one side of the triangle.

18. Compute the difference between the perimeter of a regular inscribed polygon of 90 sides and the circumference of the circle, the radius of the circle being 1 inch.

19. Show that the side of the regular inscribed polygon of n sides is $2\,R \sin \frac{180°}{n}$ and that the side of the regular circumscribed polygon is $2\,R \tan \frac{180°}{n}$, the radius of the circle being R.

Solution of Oblique Triangles by Logarithms

Points inaccessible because of swamps, hills, or other natural barriers frequently make the determination of distances, heights, or angles, by means of the right triangle, inconvenient and impracticable.



Consider, for example, the problem pictured here in which it is required to compute the distances from forts A and B to an enemy ship C. Knowing the length AB, the required distances can be computed if the measures of the angles A and B are found. We thus have an oblique triangle of which two angles and the included side are known. From this information, the required distances can be computed.

As you know, a triangle is determined if three of its six parts are known, provided at least one of these known parts is a side. Under these conditions the triangle can be solved; that is, the remaining three parts can be computed. For convenience, the treatment of the oblique triangle is subdivided into four cases.

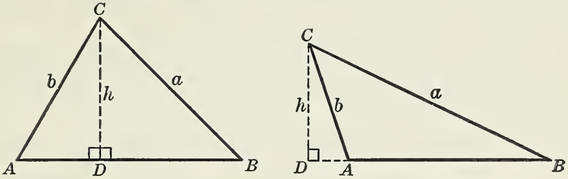
Case I. Given one side and two angles.

Case II. Given two sides and the angle opposite one of them.

Case III. Given two sides and the included angle.

Case IV. Given three sides.

Law of sines. If an altitude of any triangle be drawn, a very interesting and important relationship between the sides and the angles can be obtained. In the figure below an acute triangle is shown at the left; an obtuse triangle at the right. The same facts hold true for both triangles.



Draw altitude h from vertex C.

$$\text{Then } \frac{h}{b} = \sin A, \tag{1}$$

and

$$\frac{h}{a} = \sin B. \tag{2}$$

Dividing (1) by (2), we have

$$\frac{a}{b} = \frac{\sin A}{\sin B} \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B}. \tag{3}$$

Similarly, if the altitude from vertex A be drawn,

$$\frac{b}{\sin B} = \frac{c}{\sin C}. \tag{4}$$

From (3) and (4) we obtain

$$\boldsymbol{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}.$$

This relationship is called the *law of sines* and may be expressed in words as follows: *In any triangle the ratio of a side to the sine of its opposite angle is equal to the ratio of any other side to the sine of its opposite angle.*

Something to think about. Prove the law of sines, using a right triangle.

EXERCISES

1. Compute the radius of the circle circumscribed about an equilateral triangle whose side is 18 inches.

2. Compute the diameter of the circle circumscribed about an isosceles triangle whose base is 15 inches and whose opposite angle is 25°.

By means of the law of sines prove the following theorems:

3. If 2 angles of a triangle are equal, the sides opposite are equal.

4. If 2 sides of a triangle are equal, the angles opposite are equal.

5. (*a*) An equiangular triangle is also equilateral.

(*b*) Can the converse of this theorem be proved by the law of sines?

6. The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.

7. Two angles of a triangular plot of ground are 45° and 105°, respectively, and the included side measures 300 feet. Find the lengths of the other sides.

8. A, B, and C are three hill tops. The distance between A and B is known to be 6.12 miles and the angles CAB and CBA are observed to be $64° 53'$ and $73° 45'$, respectively. Find the distance from B to C.

9. To find the distance of a ship S from a point B on shore, a base line BA, 850 feet long, is measured along the shore. The angles SAB and SBA are observed to be $30° 28'$ and $122° 12'$, respectively. Find the ship's distance from B.

10. Two scouts stationed on shore at points A and C, 1555 feet apart, wish to determine how far from each of them a buoy (B) is located. By means of a transit, angle BAC is found to be $30° 18'$ and angle ACB $33° 53'$. Find the distances of points A and C from the buoy.

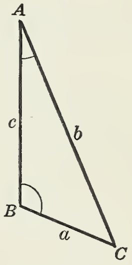
11. A tower 123.5 feet high is situated on a hill of uniform slope. The angles of depression from the top and the base of the tower to an object down the hill are 30° 13′ and 20° 10′, respectively. How far from the base of the tower is the object?

Applications of the law of sines. We are now ready to return to the oblique triangle problem mentioned on page 470.

Case I. Given one side and two angles.

Problem. The distance between forts A and B is 6.75 miles, angle $A = 23° 46'$, and angle $B = 112°$. Find the distances of the ship C from each fort, and the angle made at C by the lines of sight from the forts.

Given	*To find*
$A = 23° 46'$.	$a = ?$ mi.
$B = 112°$.	$b = ?$ mi.
$c = 6.75$ mi.	$C = ?$



Solution

$$C = 180° - (A + B)$$

$$\begin{aligned} A &= 23° 46' \\ B &= 112° \\ \hline A + B &= 135° 46' \end{aligned} \qquad \begin{aligned} 180° &= 179° 60' \\ A + B &= 135° 46' \\ \hline C &= 44° 14' \end{aligned}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

$$\therefore a = \frac{c \sin A}{\sin C}.$$

$$\log a = \log c + \log \sin A + \text{colog} \sin C.$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$\therefore b = \frac{c \sin B}{\sin C}.$$

$$\log b = \log c + \log \sin B + \text{colog} \sin C.$$

$$\begin{aligned} \log c &= .8293 \\ \log \sin A &= 9.6053 - 10 \\ \text{colog} \sin C &= .1564 \\ \hline \log a &= 10.5910 - 10 \\ &= .5910. \\ \therefore a &= 3.899. \end{aligned}$$

$$\begin{aligned} \log c &= .8293 \\ \log \sin B &= 9.9672 - 10 \\ \text{colog} \sin C &= .1564 \\ \hline \log b &= 10.9529 - 10 \\ &= .9529. \\ \therefore b &= 8.972. \end{aligned}$$

Check

In checking, it is advisable to use the table of natural functions rather than the logarithmic tables. Thus, errors which may have been made in the earlier part of the work will not be repeated in the check.

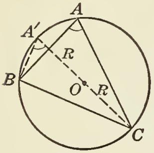
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$\frac{3.899}{.4030} = \frac{8.972}{.9272} = \frac{6.75}{.6975}.$$

$$9.68 = 9.68 = 9.68.$$

Since the measurement of side c is given correct to 3 significant figures, and angles A and B correct to the nearest minute, we shall state the answer correct to the same degree of accuracy.

$$\therefore\ C = 44° 14', a = 3.90 \text{ miles}, b = 8.97 \text{ miles}.$$



Something to think about. Using the adjoining diagram, prove the relationship between the law of sines and the diameter of the circle circumscribing the triangle:

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Arrangement of computations. So many numbers are involved in the solution of an oblique triangle by logarithms that it is exceedingly important that the computations should be arranged neatly and according to a definite plan. Carelessly written figures, poorly arranged and untidy papers, frequently lead to errors which otherwise could be avoided. It is advisable to set up the framework or outline of a solution before any logarithms are determined or calculations made. After completing the outline, the values may be filled in, the calculations made, and the answers obtained.

Notice that some of the logarithms are used more than once. Careful arrangement of your work will make such repetitions apparent and will eliminate having to look them up twice.

Outline for Case I

Given	*To find*
$A =$ $C =$ $c =$	$B =$ $a =$ $b =$
$B = 180° - (A + C).$ $A =$ $C =$ ______ $A + C =$	$180° = 179°\,60'$ $A + C =$ ______ $\therefore B =$
$\frac{a}{\sin A} = \frac{c}{\sin C}.$ $\therefore a = \frac{c \sin A}{\sin C}.$ $\log a = \log c + \log \sin A + \operatorname{colog} \sin C.$	$\frac{b}{\sin B} = \frac{c}{\sin C}.$ $\therefore b = \frac{c \sin B}{\sin C}.$ $\log b = \log c + \log \sin B + \operatorname{colog} \sin C.$
$\log c =$ $\log \sin A =$ $\operatorname{colog} \sin C =$ ______ $\log a =$ $\therefore a =$	$\log c =$ $\log \sin B =$ $\operatorname{colog} \sin C =$ ______ $\log b =$ $\therefore b =$

Check

EXERCISES

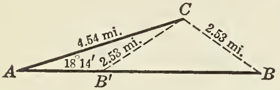
Solve the following triangles, given:

1. $A = 14°$,	$B = 68°$,	$a = 38$.
2. $A = 53°\,12'$,	$B = 40°\,30'$,	$c = 75.6$.
3. $B = 66°\,33'$,	$C = 34°\,17'$,	$a = 2.98$.
4. $B = 79°\,48'$,	$C = 80°\,36'$,	$b = 3.974$.
5. $C = 40°\,15'$,	$A = 110°\,52'$,	$a = 325.7$.

6. Two angles of a triangle are 30° and 60°, respectively. Find the ratio of the opposite sides.

Case II. The ambiguous case. Given two sides and an angle opposite one of them.

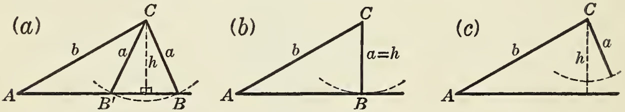
Problem. Two lighthouses, 2.53 miles apart, can be seen from a certain rock which is known to be 4.54 miles from one of them. If at the rock the angle subtended by the two lighthouses is 18° 14′, how far is the rock from the other lighthouse?



From the diagram it can be seen that the second lighthouse may be in either one of two places. The conditions in the problem do not locate the second lighthouse and it will, therefore, be necessary to solve two triangles, each one yielding a different answer to the same problem. Since we cannot tell from the problem which answer is required, this case is referred to as the "ambiguous case." You must remember, however, that if there is but one possible construction of the diagram, there will be no ambiguity in the answer.

When given values to serve as two sides and an angle opposite one of them, one, two, or even no triangle may be possible, depending upon the relations existing among the given parts. It is evident also that there will be as many solutions trigonometrically as there are triangles geometrically.

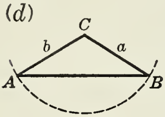
Discussion of the ambiguous case. In the diagrams below, a, b, and A represent the given parts. Observe that h, the altitude from C, can be found in terms of the given parts, from the equation $\frac{h}{b} = \sin A$; *i.e.*, $h = b \sin A$.

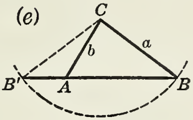


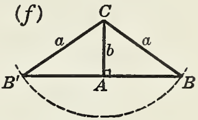
1. If A is acute, $a < b$, and $a > b \sin A$ or h, as in (a), there will be two triangles, ABC and $AB'C$, each of which has the required parts. Therefore there are two solutions.

2. If A is acute, $a < b$, and $a = b \sin A$ or h, as in (*b*), there will be one triangle, the right triangle ABC. Therefore there is one solution.

3. If A is acute, $a < b$, and $a < b \sin A$ or h, as in (*c*), no triangle can be drawn. Therefore there is no solution.







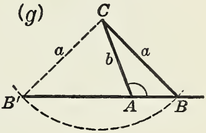
4. If A is acute and $a = b$, as in (*d*), there will be one triangle, isosceles triangle ABC. Therefore there is one solution.

5. If A is acute and $a > b$, as in (*e*), there will be one triangle, ABC. Therefore there will be one solution. Notice that triangle $AB'C$ does not satisfy the given conditions since $\angle B'AC$ is obtuse.

6. If A is a right angle and $a > b$, as in (*f*), there will be one triangle. Therefore there will be one solution.

7. If A is a right angle and $a = b$, no triangle can be drawn. Therefore there is no solution.

8. If A is a right angle and $a < b$, no triangle can be drawn. Therefore there is no solution.



9. If A is obtuse and $a > b$, as in (*g*), there will be one triangle, ABC. Therefore there will be one solution. Notice that triangle $AB'C$ does not satisfy the given conditions.

10. If A is obtuse and $a = b$, no triangle can be drawn. Therefore there is no solution.

11. If A is obtuse and $a < b$, no triangle can be drawn. Therefore there is no solution.

This discussion may be summarized in the following table and should be memorized.

The Ambiguous Case

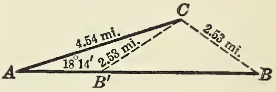
<table>
<tr><th>A</th><th>$a > b$</th><th>$a = b$</th><th colspan="2">$a < b$</th></tr>
<tr><td>obtuse</td><td>1 solution</td><td>no solution</td><td colspan="2">no solution</td></tr>
<tr><td>right</td><td>1 solution</td><td>no solution</td><td colspan="2">no solution</td></tr>
<tr><td rowspan="3">acute</td><td rowspan="3">1 solution</td><td rowspan="3">1 solution</td><td>$a < b \sin A$</td><td>no solution</td></tr>
<tr><td>$a = b \sin A$</td><td>1 solution</td></tr>
<tr><td>$a > b \sin A$</td><td>2 solutions</td></tr>
</table>

Something to think about. Show that there may be two, one, or no solution, depending upon whether log sin B is less than, equal to, or greater than zero.

Now to return to the lighthouse problem as an:

Illustrative example. Two lighthouses, 2.53 miles apart, can be seen from a certain rock which is known to be 4.54 miles from one of them. If at the rock the angle subtended by the two lighthouses is 18° 14′, how far is the rock from the other lighthouse?

Given	*To find*
$A = 18° 14'$.	$B = ?$
$a = 2.53$ mi.	$B'(AB'C) = ?$
$b = 4.54$ mi.	$C(ACB) = ?$
	$C'(ACB') = ?$
	$c(AB) = ?$
	$c'(AB') = ?$



Discussion

Since A is acute and $a < b$, we must find the value of $b \sin A$ to know how many solutions there will be. But

$b \sin A = (4.54)(.3129) = 1.42$. Therefore $a > b \sin A$ and there are 2 solutions.

Solution

Since
$$\frac{a}{\sin A} = \frac{b}{\sin B},$$
$$\sin B = \frac{b \sin A}{a}.$$
$$\therefore \log \sin B = \log b + \log \sin A + \operatorname{colog} a.*$$

$\log b = 0.6571$ $\log \sin A = 9.4954 - 10$ $\operatorname{colog} a = 9.5969 - 10$ $\log \sin B = 19.7494 - 20$ $= 9.7494 - 10.$ $\therefore B = 34° 10'.$	$B' = 180° - B.$ $180° = 179° 60'$ $B = 34° 10'$ $B' = 145° 50'$
$C = 180° - (A + B).$ $A = 18° 14'$ $B = 34° 10'$ $A + B = 52° 24'$ $180° = 179° 60'$ $A + B = 52° 24'$ $C = 127° 36'$	$C' = 180° - (A + B').$ $A = 18° 14'$ $B' = 145° 50'$ $A + B' = 164° 4'$ $180° = 179° 60'$ $A + B' = 164° 4'$ $C' = 15° 56'$

* This formula gives the value of $\sin B$ and not of B directly. Now since B is an angle of a triangle, and hence less than 180°, $\sin B$ is positive; therefore the expression $\sin B$ represents two angles, one angle B and the other angle B' (or $180° - B$). This accounts for the two solutions above. Thus, whenever an angle of a triangle is determined from its sine, two values (which are supplementary) of the angle will result and both values should be taken unless the conditions of the problem, as shown in the discussion, exclude one of them.

$$\frac{c}{\sin C} = \frac{a}{\sin A}.$$

$$c = \frac{a \sin C}{\sin A}.$$

$$\begin{aligned}\log c &= \log a + \log \sin C \\ &\quad + \operatorname{colog} \sin A.\end{aligned}$$

$$\begin{aligned}
\log a &= 0.4031 \\
\log \sin C &= 9.8989 - 10 \\
\operatorname{colog} \sin A &= \underline{0.5046} \\
\log c &= 10.8066 - 10 \\
&= 0.8066. \\
\therefore\ c &= 6.405.
\end{aligned}$$

$$\frac{c'}{\sin C'} = \frac{a}{\sin A}.$$

$$c' = \frac{a \sin C'}{\sin A}.$$

$$\begin{aligned}\log c' &= \log a + \log \sin C' \\ &\quad + \operatorname{colog} \sin A.\end{aligned}$$

$$\begin{aligned}
\log a &= 0.4031 \\
\log \sin C' &= 9.4385 - 10 \\
\operatorname{colog} \sin A &= \underline{0.5046} \\
\log c' &= 10.3462 - 10 \\
&= 0.3462. \\
c' &= 2.219.
\end{aligned}$$

Check

Use the formulas $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ and $\dfrac{b}{\sin B'} = \dfrac{c'}{\sin C'}$.

Since the measurements of sides a and b are given correct to 3 significant figures, and angle A correct to the nearest minute, we shall state the answer correct to the same degree of accuracy:

$$\begin{aligned}
B &= 34^\circ\, 10', & C &= 127^\circ\, 36', & c &= 6.41 \text{ in.} \\
B' &= 145^\circ\, 50', & C' &= 15^\circ\, 56', & c' &= 2.22 \text{ in.}
\end{aligned}$$

Illustrative examples. Find the number of solutions if:

Example 1. $A = 110^\circ$, $a = 40$, $b = 30$.

Solution

A is obtuse; $a > b$; $\therefore$ 1 solution.

Example 2. $A = 90^\circ$, $a = 16$, $b = 20$.

Solution

A is a right angle; $a < b$; $\therefore$ no solution.

Example 3. $A = 30°$, $a = 20$, $b = 40$.

Solution

A is acute; $a < b$; $b \sin A = 40(\frac{1}{2}) = 20$; $\therefore a = b \sin A$. $\therefore$ 1 solution.

Example 4. $B = 46°$, $b = 15$, $c = 20$.

Solution

B is acute; $b < c$; $c \sin B = 20(.7193) = 14.386$; $\therefore b > c \sin B$ or h. $\therefore$ 2 solutions.

EXERCISES

In the following triangles, find the number of solutions:

1.	$A = 40°$,	$a = 50$,	$b = 40$.
2.	$A = 112°$,	$a = 75$,	$b = 75$.
3.	$A = 90°$,	$a = 60$,	$b = 70$.
4.	$A = 35°$,	$a = 15$,	$b = 15$.
5.	$A = 30°$,	$a = 20$,	$b = 25$.
6.	$A = 76° 40'$,	$a = 31.2$,	$b = 38.7$.
7.	$C = 90°$,	$c = 50$,	$a = 45$.
8.	$B = 60°$,	$b = 9.56$,	$c = 10.1$.
9.	$B = 45°$,	$b = 28$,	$a = 44$.
10.	$B = 45°$,	$b = 35$,	$a = 44$.

Solve the following triangles, given:

11.	$A = 60°$,	$a = 213$,	$b = 237$.
12.	$A = 47° 19'$,	$a = 98.7$,	$b = 52.8$.
13.	$A = 136°$,	$a = 330$,	$b = 250$.
14.	$B = 30°$,	$b = 50$,	$c = 50$.
15.	$B = 60°$,	$b = 700$,	$a = 600$.
16.	$C = 21° 41'$,	$c = 84.13$,	$b = 49.27$.

17. Two stations, B and C, are on opposite sides of a bay. The distances from a third station, A, at the head of the bay, to B and C are 1030 feet and 1230 feet, respectively. If the angle ABC is known to be $49° 36'$, find the distance, BC, across the bay.

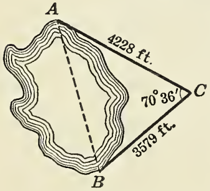
18. The diagonals of a parallelogram make an angle of $32° 18'$ with each other. If one diagonal is 43.2 inches and one side is 23.4 inches, find the other diagonal.

19. The distance from a lighthouse to the nearer of two buoys is 4000 feet. If the buoys are 3000 feet apart and the angle between the lines of sight from the lighthouse to the buoys is $20° 16'$, find the distance between the lighthouse and the other buoy. Why is there just one solution to this problem?

20. A boy who was sent to measure a triangular plot of ground ABC reports the following measurements: $A = 63° 14'$, $a = 475.0$ ft., $b = 863.2$ ft. Show that his measurements are inaccurate. If A and b are correct, what is the least value a may have, correct to the nearest tenth of a foot? Is there any maximum value for a?

Case III. Given two sides and the included angle.

Problem. Two roads which intersect at C at an angle of $70° 36'$ lead to opposite ends, A and B, of a lake. If the distances CA and CB are 4228 feet and 3579 feet, respectively, find the length of a proposed trestle from A to B, and also the angle made by the new road with each of the others.



No doubt your first thought is to use the law of cosines, $c^2 = a^2 + b^2 - 2ab \cos C$, since in this formula we know the values of all quantities except the unknown c. In view of the large numbers involved, and also because the formula is not suitable for logarithmic computation, it is not a desirable one to use here.

Therefore, before attempting a solution, we shall first derive a new set of formulas which are applicable to this case and which lend themselves to logarithmic computation.

Law of tangents. From the law of sines we have

$$\frac{a}{\sin A} = \frac{b}{\sin B}. \tag{1}$$

Applying alternation

$$\frac{a}{b} = \frac{\sin A}{\sin B}. \tag{2}$$

Applying subtraction and addition

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}. \tag{3}$$

Applying formulas on page 367

$$= \frac{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)} \tag{4}$$

$$= \cot \tfrac{1}{2}(A+B) \tan \tfrac{1}{2}(A-B). \tag{5}$$

Replacing $\cot \frac{1}{2}(A+B)$ by $\frac{1}{\tan \frac{1}{2}(A+B)}$, we obtain

$$\boldsymbol{\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}}. \tag{6}$$

This formula is known as the *law of tangents.* Stated in words: *In any triangle, the difference between two sides is to their sum as the tangent of one-half the difference between the opposite angles is to the tangent of one-half their sum.*

Note 1. If $b > a$, negative quantities can be avoided by writing the formula as follows:

$$\frac{b-a}{b+a} = \frac{\tan \frac{1}{2}(B-A)}{\tan \frac{1}{2}(B+A)}. \tag{7}$$

This form of the law of tangents can be derived exactly as above if we begin with $\frac{b}{\sin B} = \frac{a}{\sin A}$ as our first step.

Note 2. Formulas involving sides b and c as well as sides a and c may be obtained in a similar manner. They are:

$$\frac{b-c}{b+c}=\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}. \tag{8}$$

$$\frac{a-c}{a+c}=\frac{\tan\frac{1}{2}(A-C)}{\tan\frac{1}{2}(A+C)}. \tag{9}$$

Note 3. Formulas (8) and (9) may be obtained directly from (6) by what is known as *cyclic substitution.* If the letters are considered arranged in order, a, b, c; A, B, C, then (8) follows from (6) by changing the letters in order, *i.e.*, by replacing a by b, b by c, A by B, and B by C; also (9) follows from (6) by replacing b by c and B by C.

EXERCISES

Prove by the law of tangents:

1. If two angles of a triangle are equal, the sides opposite are equal.

2. If two sides of a triangle are equal, the angles opposite are equal.

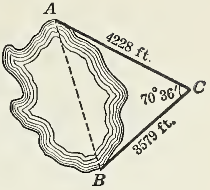
3. An equiangular triangle is also equilateral.

4. An equilateral triangle is also equiangular.

5. $\tan 15° = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ if $A = 60°$ and $B = 30°$ in triangle ABC.

We are now ready to return to our trestle road problem as an

Illustrative example. Two roads which intersect at C at an angle of 70° 36′ lead to opposite ends, A and B, of a lake. If the distances CA and CB are 4228 feet and 3579 feet, respectively, find the length of a proposed trestle from A to B, and also the angle made by the new road with each of the others.



Given	*To find*
$a = 3579$ ft.	$A = ?$
$C = 70° 36'$.	$B = ?$
$b = 4228$ ft.	$c = ?$

Solution

Since
$$\frac{\tan \frac{1}{2}(B - A)}{\tan \frac{1}{2}(B + A)} = \frac{b - a}{b + a},$$
$$\tan \tfrac{1}{2}(B - A) = \frac{(b - a) \tan \frac{1}{2}(B + A)}{(b + a)}.$$
$$\therefore \log \tan \tfrac{1}{2}(B - A) = \log (b - a) + \log \tan \tfrac{1}{2}(B + A) + \operatorname{colog} (b + a).$$

$$\begin{aligned} b &= 4228 \\ a &= \underline{3579} \\ b + a &= 7807 \\ b - a &= 649. \end{aligned}$$

$$\begin{aligned} B + A &= 180° - C. \\ 180° &= 179° 60' \\ C &= \underline{\ \ 70° 36'} \\ B + A &= 109° 24' \\ \tfrac{1}{2}(B + A) &= 54° 42'. \end{aligned}$$

$$\begin{aligned} \log (b - a) &= 2.8122 \\ \log \tan \tfrac{1}{2}(B + A) &= 10.1499 - 10 \\ \operatorname{colog} (b + a) &= \underline{6.1075 - 10} \\ \log \tan \tfrac{1}{2}(B - A) &= 19.0696 - 20 \\ &= 9.0696 - 10 \\ \therefore \tfrac{1}{2}(B - A) &= 6° 42'. \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(B + A) &= 54° 42' \\ \tfrac{1}{2}(B - A) &= \underline{\ \ 6° 42'} \\ B &= 61° 24' \\ A &= 48° 0'. \end{aligned}$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}.$$
$$c = \frac{a \sin C}{\sin A}.$$
$$\log c = \log a + \log \sin C + \operatorname{colog} \sin A.$$

$$\begin{aligned} \log a &= 3.5538 \\ \log \sin C &= 9.9746 - 10 \\ \operatorname{colog} \sin A &= \underline{0.1289} \\ \log c &= 13.6573 - 10 \\ &= 3.6573. \\ \therefore c &= 4542. \end{aligned}$$

Check

Use the formula $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Since the measurements of sides a and b are given correct to 4 significant figures, and angle C correct to the nearest minute, we shall state the answer correct to the same degree of accuracy.

$\therefore\ A = 48°\,0'$, $B = 61°\,24'$, and $c = 4542$ ft.

EXERCISES

Solve the following triangles, given:

1. $a = 625$,	$b = 525$,	$C = 85°\,35'$.
2. $a = 136$,	$c = 112$,	$B = 15°\,30'$.
3. $b = 229$,	$c = 922$,	$A = 100°$.
4. $b = 900$,	$a = 200$,	$C = 100°$.
5. $c = 625$,	$b = 431$,	$A = 140°$.
6. $b = 158$,	$c = 215$,	$A = 63°\,21'$.

7. Two sides of a triangle are 5 and 6, respectively, and include an angle of 30°. Find the third side, using the law of tangents. Check your answer by finding the third side, using the law of cosines.

8. The radius of a circle is 13.2 inches. Find the length of a chord which subtends an angle at the center of 63° 25′.

9. Two adjacent sides of a parallelogram are 81.3 feet and 128.7 feet, respectively, and include an angle of 71° 17′. Find the length of the shorter diagonal.

10. A and B, which are separated by a swamp, are sighted from a point C and the following measurements are taken: $AC = 375.7$ yards, $BC = 496.8$ yards, $\angle ACB = 59°\,28'$. Find the distance from A to B.

11. Two straight roads which make an angle of 60° 11′ with each other lead from town A to towns B and C. B on one of the roads is 68.75 miles from A, while C on the other road is 46.25 miles from A. Find the distance from B to C.

12. Two trains, traveling at rates of 45 miles and 40 miles per hour, respectively, depart from the same station at the same time on straight tracks that form an angle of 35°. How far apart will the trains be at the end of $2\frac{1}{2}$ hours?

13. The two diagonals of a parallelogram are 555 and 686, respectively, and form an angle of 58° 19′. Find the sides of the parallelogram.

14. The distances from a point R to two points (S and T) at opposite ends of a lake are 14,370 yards and 23,660 yards, respectively. If the angle subtended by the lake at R is 29° 48′, find the distance ST.

15. In order to find the distance between two markers which are invisible from each other on account of a dense wood, measurements are made from a station from which they are both visible. If the markers subtend an angle of 56° 20′ at this station and their distances from the station are 5608 and 3785 yards, respectively, find the required distance.

Case IV. Given three sides.

Problem. The frontages of a triangular plot of ground on three streets are 89.5 feet, 76.7 feet, and 110 feet. What angles do the streets make with each other?

Here we are confronted with the problem of determining angles having given the lengths of sides. You have already seen in an earlier chapter how this task can be accomplished by use of the law of cosines. But as the law of cosines formula contains addition and subtraction, it is not adapted to logarithmic computation. We shall now show how the law of cosines formula can be used to derive new formulas which are suitable for logarithmic computations and with which we shall be able to solve the above problem. These are formulas for the half angles of a triangle in terms of the sides of the triangle.

Formula for $\sin \frac{1}{2} A$.

We know that $\sin \frac{1}{2} A = \sqrt{\frac{1 - \cos A}{2}}$. (1)

Now, from the law of cosines, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}. \tag{2}$$

Subtracting each side of (2) from unity

$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2\,bc}$$

Combining $$= \frac{2\,bc - b^2 - c^2 + a^2}{2\,bc}$$

Rearranging terms $$= \frac{a^2 - (b^2 - 2\,bc + c^2)}{2\,bc} = \frac{a^2 - (b - c)^2}{2\,bc}$$

Factoring $$= \frac{(a + b - c)(a - b + c)}{2\,bc}. \tag{3}$$

Let $$2\,s = a + b + c. \tag{4}$$

Subtracting $2\,c$ from each side of (4)

$$2\,s - 2\,c = a + b + c - 2\,c$$

or $$2(s - c) = a + b - c. \tag{5}$$

Subtracting $2\,b$ from each side of (4)

$$2\,s - 2\,b = a + b + c - 2\,b$$

or $$2(s - b) = a - b + c. \tag{6}$$

Substituting (5) and (6) in (3)

$$1 - \cos A = \frac{2(s - c)\cancel{2}(s - b)}{\cancel{2}\,bc}. \tag{7}$$

Dividing each side of (7) by 2

$$\frac{1 - \cos A}{2} = \frac{\cancel{2}(s - c)(s - b)}{\cancel{2}\,bc}. \tag{8}$$

Substituting (8) into (1), we have

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{bc}}.$$

By cyclic substitution we obtain

$$\sin \tfrac{1}{2} B = \sqrt{\frac{(s - c)(s - a)}{ca}} \quad \text{and} \quad \sin \tfrac{1}{2} C = \sqrt{\frac{(s - a)(s - b)}{ab}}.$$

Formula for $\cos \frac{1}{2} A$.

We know that $\cos \frac{1}{2} A = \sqrt{\dfrac{1 + \cos A}{2}}$. (1)

Now, from the law of cosines, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}. \tag{2}$$

Adding each side of (2) to unity,

$$1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2\,bc}$$

Combining $= \dfrac{2\,bc + b^2 + c^2 - a^2}{2\,bc}$

Rearranging terms $= \dfrac{(b^2 + 2\,bc + c^2) - a^2}{2\,bc}$

$$= \frac{(b + c)^2 - a^2}{2\,bc}$$

Factoring $= \dfrac{(b + c + a)(b + c - a)}{2\,bc}$. (3)

Let $2\,s = a + b + c$. (4)

Subtracting $2\,a$ from each side of (4)

$$2\,s - 2\,a = a + b + c - 2\,a$$

or $2(s - a) = b + c - a$. (5)

Substituting (4) and (5) into (3)

$$\therefore 1 + \cos A = \frac{2\,s \cdot \cancel{2}(s - a)}{\cancel{2}\,bc}. \tag{6}$$

Dividing each side of (6) by 2

$$\frac{1 + \cos A}{2} = \frac{\cancel{2}\,s(s - a)}{\cancel{2}\,bc}. \tag{7}$$

Substituting (7) into (1), we have

$$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s - a)}{bc}}.$$

By cyclic substitution we obtain

$$\cos \tfrac{1}{2} B = \sqrt{\frac{s(s - b)}{ca}} \quad \text{and} \quad \cos \tfrac{1}{2} C = \sqrt{\frac{s(s - c)}{ab}}.$$

Formula for $\tan \frac{1}{2} A$.

Since $\tan x = \dfrac{\sin x}{\cos x}$ $\quad \tan \frac{1}{2} A = \dfrac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A}$.

Substituting the values just obtained for $\sin \frac{1}{2} A$ and $\cos \frac{1}{2} A$

$$\tan \tfrac{1}{2} A = \frac{\sqrt{\dfrac{(s-b)(s-c)}{bc}}}{\sqrt{\dfrac{s(s-a)}{bc}}}.$$

Simplifying, we obtain

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

By cyclic substitution we obtain

$$\tan \tfrac{1}{2} B = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \quad \text{and} \quad \tan \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

Another form for $\tan \frac{1}{2} A$.

From above

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Multiplying numerator and denominator by $(s - a)$

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}}$$

Simplifying $\quad = \dfrac{1}{s-a}\sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$. (1)

Let $\sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}} = r.$ (2)

Substituting (2) in (1), we obtain

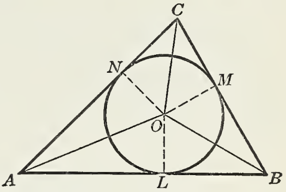
$$\tan \tfrac{1}{2} A = \frac{r}{s-a}.$$

By cyclic substitution we obtain

$$\tan \tfrac{1}{2} B = \frac{r}{s-b} \quad \text{and} \quad \tan \tfrac{1}{2} C = \frac{r}{s-c}.$$

The value of r as given in equation 2 on page 490 is actually the value of the radius of the circle inscribed in triangle ABC, as the following proof will illustrate.

Formula for the radius of the circle inscribed in a triangle. In the adjoining figure, angles A, B, and C are bisected; OL, OM, and ON (radii of the inscribed circle) are perpendicular to AB, BC, and CA, respectively.



$$\tan \angle OAL = \tan \tfrac{1}{2} A = \frac{OL}{AL}. \quad (1)$$

Representing the perimeter by $2s$, we have

$$\begin{aligned} 2s &= NA + AL + LB + BM + MC + CN \\ &= 2AL + 2BM + 2MC \qquad \left[\begin{aligned}\text{Since } NA &= AL \\ LB &= BM \\ CN &= MC\end{aligned}\right] \\ &= 2AL + 2(BM + MC) \\ &= 2AL + 2a. \\ \therefore s &= AL + a \end{aligned}$$

or $$AL = s - a. \qquad (2)$$

Substituting (2) into (1), we obtain

$$\tan \tfrac{1}{2} A = \frac{OL}{s-a}. \qquad (3)$$

But on page 490 we showed that

$$\tan \tfrac{1}{2} A = \frac{r}{s-a} \qquad (4)$$

in which $$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Now comparing (3) with (4), we observe that

$$OL = r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Thus in the formulas for the tangents of the half angles of a triangle, r is the radius of the inscribed circle.

Something to think about. The square roots of a positive number have the same absolute value but are opposite in sign. Why do we use only the positive square root in all the half-angle formulas of a triangle.

We are now ready to solve the problem on page 487 as an

Illustrative example. The frontages of a triangular plot of ground on three streets are 89.5 feet, 76.7 feet, and 110 feet. What angles do the streets make with each other?

Given	*To find*
$a = 89.5.$	$A = ?$
$b = 76.7.$	$B = ?$
$c = 110.$	$C = ?$

Solution

$a = 89.5$	$\therefore\ s = 138.1,$
$b = 76.7$	$s - a = 48.6,$
$c = 110$	$s - b = 61.4,$
$2s = 276.2$	$s - c = 28.1.$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$\log r = \tfrac{1}{2}[\log(s-a) + \log(s-b) + \log(s-c) + \operatorname{colog} s].$$

$$\begin{aligned}
\log(s-a) &= 1.6866\\
\log(s-b) &= 1.7882\\
\log(s-c) &= 1.4487\\
\operatorname{colog} s &= \underline{7.8598 - 10}\\
&\ 12.7833 - 10\\
&\ 2)\underline{2.7833}\\
\log r &= 1.3917
\end{aligned}$$

$$\tan \tfrac{1}{2} A = \frac{r}{s-a}.$$
$$\log \tan \tfrac{1}{2} A = \log r - \log (s-a).$$
$$\begin{aligned} \log r &= 11.3917 - 10 \\ \log (s-a) &= \underline{1.6866} \\ \log \tan \tfrac{1}{2} A &= 9.7051 - 10 \\ \tfrac{1}{2} A &= 26^\circ\, 53'. \\ A &= 53^\circ\, 46'. \end{aligned}$$

$$\tan \tfrac{1}{2} B = \frac{r}{s-b}.$$
$$\log \tan \tfrac{1}{2} B = \log r - \log (s-b).$$
$$\begin{aligned} \log r &= 11.3917 - 10 \\ \log (s-b) &= \underline{1.7882} \\ \log \tan \tfrac{1}{2} B &= 9.6035 - 10 \\ \tfrac{1}{2} B &= 21^\circ\, 52'. \\ B &= 43^\circ\, 44'. \end{aligned}$$

$$\tan \tfrac{1}{2} C = \frac{r}{s-c}.$$
$$\log \tan \tfrac{1}{2} C = \log r - \log (s-c).$$
$$\begin{aligned} \log r &= 11.3917 - 10 \\ \log (s-c) &= \underline{1.4487} \\ \log \tan \tfrac{1}{2} C &= 9.9430 - 10 \\ \tfrac{1}{2} C &= 41^\circ\, 15'. \\ C &= 82^\circ\, 30'. \end{aligned}$$

Check

$$\begin{aligned} A + B + C &= 180^\circ. \\ 53^\circ\, 46' + 43^\circ\, 44' + 82^\circ\, 30' &= 180^\circ. \\ 180^\circ &= 180^\circ. \end{aligned}$$

Since the measurements of sides a, b, and c are given correct to 3 significant figures, it is sufficient to state the answer correct to the nearest minute.

$\therefore\ A = 53^\circ\, 46',\quad B = 43^\circ\, 44',\quad C = 82^\circ\, 30'.$

EXERCISES

Solve the following triangles, given:

1. $a = 13,\quad b = 14,\quad c = 15.$
2. $a = 7,\quad b = 24,\quad c = 25.$
3. $a = 900,\quad b = 4000,\quad c = 4100.$
4. $a = 34.16,\quad b = 28.17,\quad c = 16.07.$

5. $a = 2376, \quad b = 4238, \quad c = 3116.$

6. $a = .2265, \quad b = .2721, \quad c = .3310.$

7. The sides of a triangle are 43.2, 34.7, and 26.8 respectively. Find the largest angle.

8. Find the central angle subtended by a chord 5.56 yards long in a circle of radius 14.72 yards.

9. The sides of a triangle are 12.6, 14.4, and 16.8 respectively. Find the smallest angle and the length of the altitude to the longest side.

10. Two adjacent sides of a parallelogram are 102.7 and 84.3, respectively, and the length of the diagonal joining their ends is 134.2. Find the angles of the parallelogram.

11. A triangular plot of ground having sides equal to 100 feet, 175 feet, and 150 feet, respectively, is to be laid off so that its frontage on a straight state highway is 100 feet. Compute the two angles necessary to accomplish this.

12. Find the radius of the circle inscribed in a triangle, whose sides are 11, 21, and 31 inches respectively.

13. A cylindrical gas tank is to be constructed on a triangular plot of ground whose sides are 80, 90, and 100 feet. Find the largest possible radius of the tank.

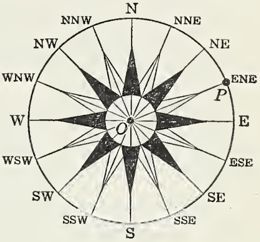
14. If Albany is assumed to be 150 miles due north of New York City, while city B is 100 miles from Albany and 125 miles from New York City, how many miles is city B north of New York City?

Plane Sailing

A ship sailing on an ocean, a train speeding from one city to another, or even a cross country runner, never actually travels along a straight line. If the earth were flat, it would be possible to travel along straight lines; but the earth is not flat, and therefore there are really no straight lines on its surface. However, when comparatively small distances on the earth's surface are considered, the curvature of the earth may be neglected; the earth in that case can be considered as a plane. It is then that the principles of plane trigonometry may be applied.

Plane sailing is that branch of navigation in which the earth's surface is considered as a plane.

The *compass* is a device for locating directions in a plane. The directions indicated on the compass are usually referred to as "bearings." In the accompanying diagram of a compass, you will observe that the circumference is divided into 16 main divisions or directions, each of which corresponds to $22\frac{1}{2}°$.



Many interesting problems dealing with plane sailing will be found among the examples in the following exercise.

Observe that ENE, and N 67° 30′ E, indicate the same point (P) on the compass. Since directions on the compass are frequently referred to as "bearings," the "bearing" of P from O is ENE, or N 67° 30′ E, or E 22° 30′ N. Thus to locate a point, given its "bearing," we measure the number of degrees from the first letter as a starting point in the direction indicated by the second letter.

MISCELLANEOUS EXERCISES

Group A

Solve the following triangles, given:

1. $a = 396, \quad b = 645, \quad C = 90°.$

2. $b = 32.7, \quad a = 43.4, \quad A = 60° 7'.$

3. $B = 47° 15', \quad C = 24° 57', \quad b = 147.$

4. $a = 52, \quad b = 62, \quad c = 72.$

5. $a = 847, \quad b = 573, \quad C = 63° 55'.$

Prove for any triangle:

6. $\tan \frac{1}{2} A \tan \frac{1}{2} B = \dfrac{a + b - c}{a + b + c}.$

7. $\dfrac{\sin A}{a} = \dfrac{\sin A + 2 \sin B}{a + 2b}.$

8. Two angles of a triangle are 25° and 50°. Find the ratio of the opposite sides.

9. The sides of a right triangle are 5, 12, and 13. Find (*a*) one acute angle, (*b*) the altitude upon the hypotenuse, (*c*) the segments of the hypotenuse made by the altitude.

10. The base of a triangle is 7.56 inches and the base angles are 20° and 100°. Find the other sides and the altitude to the base.

11. From two coast stations A and B, 4280 yards apart, a ship C is observed at sea. The angles ABC and BAC are measured at the same instant and are observed to be 50° 12′ and 69° 36′ respectively. Find the distance of the ship from A.

12. Two sides of a triangle are 4.75 inches and 6.83 inches, respectively, and include an angle of 68° 36′. Find the third side.

13. To find the length AB across a stream, a base line AC, 340 feet long, is taken along the shore. The angles CAB and ACB are measured and found to be 80° 45′ and 63° 40′ respectively. Find the distance from A to B.

14. In order to find the distance between two points A and B, which are separated by a swamp, the following measurements were taken from a station selected at C: $CA = 4820$ yards, $CB = 4370$ yards, $\angle BCA = 67° 33'$. Find the distance AB.

15. The sides of a triangle are 9, 40, and 41 respectively. Find the smallest angle.

16. A and B are two points on opposite sides of a mountain and C is a point visible from both A and B. It is found that the distances from C to A and B are 2347 feet and 3148 feet respectively and that angle ACB is $49° 11'$. Find the distance from A to B.

17. A 14-foot pole is placed against an embankment so that one end reaches 7 feet on the ground from the base of the embankment, while the other end reaches 9 feet up its face. Find the slope of the embankment if the pole lies in a plane perpendicular to its base edge.

18. What part of the circumference of a compass lies between (*a*) N and NE? (*b*) N and NNE? (*c*) N and ENE? (*d*) W and SW? (*e*) S and ESE?

19. From a station A, the two ends of a lake bear N 53° W and N 76° E. The distances from A to the ends are respectively 3278 feet and 3427 feet. Find the length of the lake.

20. A ship sails ENE at the rate of 25 knots per hour. Find the rate at which it is moving north; also the rate at which it is moving east.

21. At a certain point a captain sights a lighthouse lying due east of him, and after sailing north 8 miles, he observes the lighthouse to be in the direction S. 60° 40′ E. How far was the ship from the lighthouse at the time of the second observation?

22. A ship sailing due west at the rate of 10.43 miles per hour passes a lighthouse due south of it at 10 P.M. At 12.15 P.M. the bearing of the same lighthouse was S 40° 30′ E. Find the ship's distance from the lighthouse at the time of the second observation.

Group B

Solve the following triangles:

23. $A = 34° 25'$, $c = 726.4$, $C = 90°$.

24. $a = 466.4$, $b = 474.6$, $A = 75°$.

25. $a = 45.42$, $B = 29° 17'$, $C = 90°$.

26. $C = 68° 15'$, $B = 45° 18'$, $c = 45.67$.

27. $a = .1234$, $b = .2768$, $c = .3001$.

Prove for any triangle:

28. $a \cos \frac{1}{2} B \cos \frac{1}{2} C = s \sin \frac{1}{2} A$.

29. $a = b \cos C + c \cos B$. **30.** $b = a \cos C + c \cos A$.

31. The base of a triangle is 8.43 inches and the base angles are 25° 15′ and 100° 25′. Find the altitude to the base.

32. The angles of a triangle are 18°, 54°, and 108°. Find the ratios of the sides.

33. Two sides of a triangle are 27 inches and 36 inches, and the angle opposite the 27-inch side is 24°. Find the possible values for the third side.

34. The sides of a triangle are 6, 12, and 11. Find the ratio of the smallest to the largest angle.

35. A tunnel is to be dug through a hill connecting two points on opposite sides. If the distances of the points from a station are 3769 yards and 1006 yards, respectively, and the angle subtended at the station by the distances is 59° 36′, find the length of the proposed tunnel.

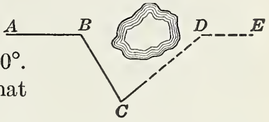
36. A watch tower is located on the summit of a hill which has a uniform incline of 28° 14′ to the horizontal. At a point 40.28 feet from the base of the tower, measured down the hill, the angle subtended by the tower is 21° 10′. Find the height of the tower.

37. Two straight railroad tracks meet at an angle of 78° 48′. If two trains traveling at 37 and 30 miles an hour, respectively, start from the junction at the same time, how far apart will they be after 45 minutes?

38. A side of a regular pentagon is 10 inches. Find the length of a diagonal.

39. The sides of a triangle are 14 inches, 16 inches, and 18 inches. Find the length of the median to the longest side.

40. A surveyor, wishing to continue the straight line AB on the other side of a pond, detours around the obstruction as follows: $\angle ABC = 125°$, $BC = 165$ feet, $\angle C = 80°$. How long must he make CD and what angle must DE make with CD?



41. Two forces, one of 152 pounds and the other of x pounds, act at an angle of $71° 25'$. If their resultant force makes an angle of $38° 11'$ with the 152-pound force, find x. (The resultant of two forces is represented by the longer diagonal of the parallelogram whose sides represent 152 and x, the included angle being $71° 25'$.)

42. The angle of elevation of an airplane from point A due west of it is 58°. From point B on the same level as A but due east of the airplane the angle of elevation is 45°. If the distance between A and B is 1200 feet, find the height of the airplane.

43. Two lights, one 45 miles away and the other 38 miles away, are visible from a ship. If the first light bears SSE (S $22\frac{1}{2}°$ E) while the second WSW (S $67\frac{1}{2}°$ W), how far apart are the lights?

44. From a ship a lighthouse was observed to be N 43° E. After the ship had sailed $4\frac{1}{2}$ miles due south the lighthouse was N $32° 15'$ E. Find the ship's distance from the lighthouse when the first observation was made; the second.

Group C

Solve the following triangles, given:

45. $a = 15\sqrt{3},\ B = 90°,\ C = 65°.$

46. $a = \sqrt{15},\ b = \sqrt{23},\ c = \sqrt{29}.$

47. $a = \sqrt{5} + 1,\ b = \sqrt{5} - 1,\ c = 2\sqrt{3}.$

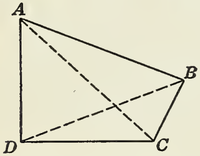
48. Prove that $\dfrac{\cos \frac{1}{2}(A - B)}{\sin \frac{1}{2} C} = \dfrac{a + b}{c}$ for all triangles.

49. The angles of a triangle are as 3 : 4 : 5. Find the ratio of the shortest side to the longest.

50. Two ships start at the same time from the same port and sail in straight courses at an angle of 75° 18′ to each other. After sailing for $4\frac{1}{2}$ hours at 15.5 and 18.5 knots per hour, the ships turn and sail directly toward each other. Find the angles which the last course makes with the original courses.

51. The equal sides of an isosceles triangle are each represented by a and the altitude on the other side by h. Write formulas expressing each of the angles of the isosceles triangle in terms of a and h.

52. In order to determine the distance between two inaccessible points, A and B, the following measurements were taken from points C and D in the same horizontal plane with A and B:



$$\begin{aligned} CD &= 6250 \text{ feet}, \\ \angle BDC &= 23^\circ 46', \\ \angle BCD &= 116^\circ 26', \\ \angle ADC &= 91^\circ 28', \\ \angle ACD &= 42^\circ 49'. \end{aligned}$$

Find AB. How can you tell from the conditions of the problem that the answer must be greater than 6250 feet?

53. A and B are two observation posts situated on the same straight level road on which is located a tower, both posts being on the same side of the tower. At A the angle of elevation of the top of the tower is $m°$; at B, which is c yards nearer the tower, the angle of elevation of the top is $n°$. Show that the height of the tower (h) is $h = \dfrac{c \sin n° \sin m°}{\sin (n° - m°)}$.

54. A battleship is 20 miles due south of a lighthouse when it sets out on a course that is N 22° 30′ E. If objects can be seen at a distance of 10 miles, how far must the ship go before the lighthouse can be seen? If the battleship can make 25 miles an hour, for how long will the lighthouse be in view?

55. An engineer of a train going due west at a uniform rate observes a signal tower 1 mile distant bearing due north at 1 P.M.; 5 minutes later the tower bears N 67° 51′ E. Find the rate of the train and the bearing of the tower at 2 P.M.

56. Steamer *A* is 160 miles NNE and steamer *B* is 170 miles NW from a certain station. If each ship has wireless apparatus with a range of 250 miles, can the ships communicate directly with each other?

The Triangle Conquered

The formulas necessary for the solution of a triangle were practically all developed before 1600, and then further improved upon with the discovery of logarithms. The publications of Vieta in 1579 contained some very remarkable contributions in the field of trigonometry. Here we find the first systematic development of the methods of solving triangles with the aid of the six trigonometric functions.

It is interesting to note that while the law of tangents first appears in the writings of Vieta, the law of cosines was known to Euclid in 300 B.C.

CUMULATIVE REVIEW

Chapters XII, XIII, and XIV

1. Which of these statements are true? Which are false?

(*a*) $(27\,x^5)^{\frac{1}{3}} = 3\,x\sqrt[3]{x^2}$.

(*b*) $2^{\frac{3}{4}} \times 2^{\frac{4}{3}} = 2$.

(*c*) $729 \times 81 = 3^6 \times 3^4 = 9^{10}$.

(*d*) $\log\left(\dfrac{a}{b}\right) = \dfrac{\log a}{\log b}$.

(*e*) If $\log x = .4639$, then $\log 10\,x = 1.4639$.

(*f*) If the diagonal and base of a rectangle are 8.2 inches and 7.5 inches, respectively, their included angle is 45°.

(*g*) If the radius of a circle is 10, then the side of a regular inscribed pentagon is also 10.

(*h*) A ship sails directly northwest at the rate of 20 miles an hour. At the end of one hour the ship will be less than 20 miles north of the starting point.

(*i*) In the right triangle in which $a = 5$, $b = 12$, and $c = 13$, $\log \sin B > \log \sin A$.

(*j*) In the triangle in which $A = 132°$, $a = 56$ feet, and $b = 37$ feet, there may be two solutions.

2. Complete each of the following statements:

(*a*) The value of $27^{\frac{1}{3}}$ divided by 3^{-3} is ? .

(*b*) When $a = 5$, the value of $\frac{\sqrt{a}}{\sqrt[6]{a^5}} \times a^{-\frac{2}{3}}$ is ? .

(*c*) Log. .06388 = ?

(*d*) If $\tan x = .6158$, then $x = ?$ (correct to the nearest minute).

(*e*) Colog 5.38 = ? .

(*f*) If each of the equal sides of an isosceles triangle is 40 and the altitude to the third side is 16, each of the equal angles, correct to the nearest degree, is ? .

(*g*) When a 50-foot vertical pole casts a horizontal shadow 30 feet long, the angle of elevation of the sun, correct to the nearest degree, is ? .

(*h*) In triangle ABC, if $a = 4$, $b = 8$, and $C = 60°$, then $c = ?$

(*i*) In triangle ABC, if $a = 4$, $b = 5$, and $c = 6$, then $\cos A = ?$

(*j*) If a, b, A, and B are parts of oblique triangle ABC, the formula expressing a in terms of b and functions of A and B is ? .

3. Find the value of $\sqrt[5]{1.4}$ correct to the nearest hundredth.

4. Draw the graph of $x = 3^y$, from $y = -4$ to $y = +4$.

5. Find the value of: $\dfrac{(6.27)^6 \times \sqrt[3]{.431}}{98.32}$.

6. The radius (R) of a sphere in terms of its volume (V) is given by the relationship: $R = \sqrt[3]{\dfrac{3V}{4\pi}}$. Find R correct to the nearest hundredth if $V = 9.46$ and $\pi = 3.14$.

7. If \$2340 is deposited in a bank at 6%, interest compounded annually, in how many years will it amount to \$4680?

8. In right triangle ABC ($C = 90°$), $A = 24° 15'$, and $a = 136.3$ feet. Find, by logarithms, b and c.

9. Find the perimeter of a regular pentagon inscribed in a circle whose radius is 16 inches.

10. Wishing to find the width of a pond, a scout picked three trees, A, B, and C, choosing A and B on opposite edges of the pond, and made the following measurements: $\angle BAC = 51° 17'$, $AC = 1377$ feet, $\angle ACB = 43° 23'$. Find AB, the width of the pond.

11. Solve the triangle ABC, given $a = 62.6$, $b = 58.8$, $c = 45.0$.

12. Town A lies 72 miles due north of town B. A third town C lies farther west than A and B. The distance from A to C is 55 miles and from B to C is 62 miles.

(a) Find angle ABC.
(b) In what direction does C lie from B?

CHAPTER XV. FINDING AREAS OF TRIANGLES

When you can measure what you are speaking about and express it in numbers, you know something about it. — LORD KELVIN.

In one of the much-used formulas of plane geometry, $K = \frac{1}{2}bh$, K, b, and h represent respectively the area, base, and altitude of a triangle. In theoretical work this formula has a wide variety of applications, but in practical work it is not so useful because it is much easier to measure the sides and angles of a triangle than it is to measure one of its altitudes. Therefore it is advisable to derive new formulas that express the area of a triangle in terms of its sides and angles. Incidentally the formula $K = \frac{1}{2}bh$ is our starting point in the derivation of these new formulas.

Do you remember how to construct a triangle with ruler and compass? Explain how you would construct a triangle with sides 2 inches and 3 inches and the angle between them 60°. How would you construct a triangle with sides 1 inch, $1\frac{1}{2}$ inches, and 2 inches? a triangle two of whose angles are 45° and 60° and the side between these angles 1 inch in length?

From this work you can see that if three parts of a triangle are known, at least one of which is a side, the triangle can be constructed and its area determined. We shall therefore consider the area of a triangle under each of the following cases:

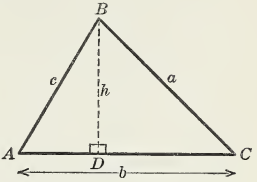
Case I: Given two sides and the included angle.

Case II: Given a side and any two angles.

Case III: Given the three sides.

Case I. Given two sides and the included angle.

Let K represent the area of triangle ABC and b, c, and A its given parts. Draw $BD \perp AC$.



$$K = \tfrac{1}{2}\,bh. \qquad (1)$$

But $$\frac{h}{c} = \sin A\,;$$

$$\text{i.e., } h = c \sin A. \qquad (2)$$

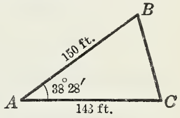
Substituting (2) in (1), we have

$$\mathbf{K = \tfrac{1}{2}\,bc \sin A.}$$

Similarly, by drawing the other altitudes, we obtain

$$\mathbf{K = \tfrac{1}{2}\,ac \sin B} \quad \text{and} \quad \mathbf{K = \tfrac{1}{2}\,ab \sin C.}$$

Illustrative example. Find the area of the triangle in the adjoining diagram if $b = 143$ ft., $c = 150$ ft., and $A = 38° 28'$.



Solution

$$K = \tfrac{1}{2}\,bc \sin A.$$

$$\begin{aligned}
\log K &= \log b + \log c + \log \sin A + \text{colog } 2.\\
\log b &= 2.1553\\
\log c &= 2.1761\\
\log \sin A &= 9.7938 - 10\\
\text{colog } 2 &= \underline{9.6990 - 10}\\
\log K &= 23.8242 - 20\\
&= 3.8242.\\
\therefore\ K &= 6671.
\end{aligned}$$

Since the measurements of sides b and c are given correct to 3 significant figures, we shall state the answer correct to the same degree of accuracy.

The area of triangle ABC is 6670 sq. ft.

EXERCISES

Find the area of each of the following triangles, given:

1. $b = 72$, $c = 23$, $A = 47°$.

2. $b = 49$, $c = 56$, $A = 74° 14'$.

3. $b = 128$, $c = 157$, $A = 100° 10'$.

4. $a = 123.4$, $c = 123.4$, $B = 39° 19'$.

5. $a = 256.8$, $c = 256.8$, $B = 110° 18'$.

6. $a = 3.547$, $b = 2.475$, $C = 53° 45'$.

7. $a = 117.0$, $b = 98.77$, $C = 75° 57'$.

8. $a = 782.4$, $b = 568.9$, $C = 44° 54'$.

9. $b = 322.4$, $c = 453.2$, $A = 80° 15'$.

10. $a = 155.6$, $c = 155.6$, $B = 125° 40'$.

11. Two streets intersect at an angle of 63° 30′. A triangular corner lot has a frontage of 125 feet on one street and 156 feet on the other. What is the area of the lot in square feet?

12. Two sides of a triangle are 168 feet and 175 feet and its area is 7350 square feet. Find the angle between the given sides.

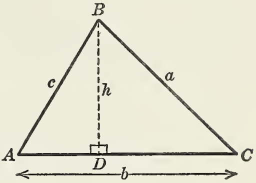
13. The area of triangular lot ABC is 4350 square feet. If angle $A = 75°$ and $AB = 100$ feet, find AC.

14. Prove that the area of a parallelogram is equal to the product of two of its adjacent sides multiplied by the sine of their included angle.

15. Two sides of a parallelogram are 178.8 feet and 194 4 feet, and their included angle is 52° 28′. Find the area of the parallelogram.

Case II. Given a side and any two angles.

Let K represent the area of the triangle ABC and a, B, and C its given parts. Draw $BD \perp AC$. Angle A can be found, since $A = 180° - B - C$.



$$K = \tfrac{1}{2} bh. \qquad (1)$$

But
$$\frac{h}{a} = \sin C;$$

$$\text{i.e., } h = a \sin C, \qquad (2)$$

and
$$\frac{b}{\sin B} = \frac{a}{\sin A}; \text{ i.e., } b = \frac{a \sin B}{\sin A}. \qquad (3)$$

Substituting (2) and (3) in (1), we have

$$K = \frac{1}{2} \cdot \frac{a \sin B}{\sin A} \cdot a \sin C,$$

or
$$\boldsymbol{K = \frac{a^2 \sin B \sin C}{2 \sin A}}.$$

Similarly by drawing the other altitudes, we obtain

$$\boldsymbol{K = \frac{b^2 \sin C \sin A}{2 \sin B}} \quad \text{and} \quad \boldsymbol{K = \frac{c^2 \sin A \sin B}{2 \sin C}}.$$

Something to think about. Show that the formula

$$K = \frac{a^2 \sin B \sin C}{2 \sin A} \text{ can be written } K = \frac{a^2 \sin B \sin C}{2 \sin (B + C)}.$$

EXERCISES

Find the area of each of the following triangles, given:

1. $a = 15$, $B = 55°$, $C = 55°$.
2. $a = 37.2$, $B = 36° 45'$, $A = 86° 55'$.
3. $b = 35.6$, $A = 58° 30'$, $C = 46° 20'$.
4. $c = 150$, $A = 105° 12'$, $B = 40° 36'$.
5. $a = 128.6$, $B = 76° 45'$, $C = 97° 40'$.
6. In a triangular field one side, 900 feet long, makes angles of $71° 35'$ and $76° 52'$ with the other sides. Find the area of the field.

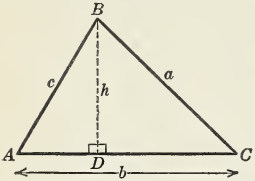
7. The area of an isosceles triangle is 1070 square feet. Find its base, if one of its equal base angles is 43° 50′.

8. In parallelogram $ABCD$, $AB = 28.4$ inches, $\angle CAB = 31° \; 18'$, and $\angle ACB = 42° \; 12'$. Find the area of the parallelogram.

9. In a triangular field ABC, AB runs N 10° E, 155.5 feet, BC runs S 51° E, and CA runs N 80° W. Find the area of the field.

Case III. Given the three sides.

Let K represent the area of the triangle ABC and a, b, c its given parts. Draw $BD \perp AC$.



$$K = \tfrac{1}{2}\,bh. \tag{1}$$

But $$\frac{h}{c} = \sin A;$$

i.e., $$h = c \sin A. \tag{2}$$

Substituting (2) in (1), we have

$$K = \tfrac{1}{2}\,bc \sin A. \tag{3}$$

But $$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$$

and $$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

and $$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}. \qquad \text{Why?}$$

$$\therefore \sin A = 2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$$

$$= 2\sqrt{\frac{s(s-a)(s-b)(s-c)}{b^2c^2}}$$

$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}. \tag{4}$$

Substituting (4) in (3),

$$K = \tfrac{1}{2}\,bc \cdot \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

$$\therefore \boldsymbol{K = \sqrt{s(s-a)(s-b)(s-c)}}.$$

Something to think about.

1. With the aid of the formula just established, prove that $h_b = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$.

2. What would be the corresponding formulas for h_c and h_a?

3. In computing the area of a triangle how would you decide upon the formula to be used?

EXERCISES

Find the area of each of the following triangles, given:

1. $a = 5$, $b = 12$, $c = 13$.

2. $a = 10$, $b = 12$, $c = 14$.

3. $a = 82.7$, $b = 96.8$, $c = 100$.

4. $a = 555$, $b = 555$, $c = 555$.

5. $a = 156.8$, $b = 175.6$, $c = 148.1$.

6. What is the area of a triangular slab whose sides measure 1.86 inches, 1.43 inches, and 2.05 inches?

7. The sides of a triangle whose area is $6\sqrt{6}$ are in the ratio $10:12:14$. Find the sides.

8. Find the area of quadrilateral $ABCD$ if $AB = 5$ inches, $BC = 12$ inches, $CD = 15$ inches, $AD = 14$ inches, and diagonal $AC = 13$ inches.

MISCELLANEOUS EXERCISES

Group A

Find the area of each of the following triangles, given:

1. $b = 48$, $c = 73$, $A = 56° 23'$.

2. $a = 145$, $B = 58° 12'$, $C = 79° 19'$.

3. $c = 109$, $b = 6.6$, $a = 108$.

4. $b = 43.1$, $B = 99° 28'$, $C = 46° 16'$.

5. Find the area of a regular decagon having a side equal to 9.3 inches.

6. The sides of a field $ABCD$ are: $AB = 21$ feet, $BC = 29$ feet, $CD = 99$ feet, $DA = 101$ feet, and the distance from A to C is 20 feet. Find the area of the field.

7. Find the area of a parallelogram if its adjacent sides are 65 yards and 59 yards and their included angle is $56° 28'$.

8. A lot in the shape of a rhombus measures 23.6 feet on a side. One of its angles is $66° 50'$. Find its area.

9. The sides of a triangular grass plot are 28 feet, 45 feet, and 53 feet. Find the radius and area of the largest circular flower bed that can be planted in the plot.

Group B

Find the area of each of the following triangles, given:

10. $c = 12770$, $b = 226$, $a = 12760$.

11. $a = 24.63$, $c = 35.81$, $B = 63° 23'$.

12. $b = 243.3$, $B = 91° 15'$, $A = 39° 38'$.

13. Prove that in any triangle ABC,

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

14. Prove that the area of a parallelogram is equal to one-half the product of its two diagonals and the sine of their included angle.

15. Find the area of a parallelogram whose diagonals, 108 feet and 216 feet long, intersect at an angle of $69° 35'$.

16. A lot in the shape of a rhombus measures 45 feet on a side and has an area of 1900 square feet. Find the angles in which the sides of the lot intersect.

17. Find the difference between the areas of the circumscribed and inscribed regular pentagons if the radius of the circle is 12 inches.

Group C

18. Prove that the area of any quadrilateral is equal to one-half the product of its two diagonals and the sine of their included angle.

19. The diagonals of a quadrilateral are 418.7 feet and 576.3 feet, and intersect at an angle of 26° 37′. Find the area of the quadrilateral.

20. A quadrangular lot has frontages of 100 feet and 120 feet on two streets which intersect at an angle of 68°. If the remaining two boundary lines are perpendicular to the two streets, what is the area of the lot?

21. Prove that the area of any triangle (ABC) is

$$bc \sin \frac{A}{2} \cos \frac{A}{2}.$$

22. If b is one side of a regular polygon of n sides, prove that the area of the polygon is $\frac{nb^2}{4} \cot \frac{180°}{n}$.

23. Two points A and B are 4.5 miles apart and are situated so that A is due north of B. A third point C bears N 76° 45′ E from A and from B it bears N 28° 35′ E. Find the number of square miles in triangle ABC.

24. Find the area of a quadrangular field $ABCD$, having given: AB runs N 32° E for 168 feet, BC runs S 37° E for 155 feet, CD = 120 feet, DA = 100 feet.

Trigonometry Comes into Its Own

As noted previously, the work of Regiomontanus (1436–1476) marked the beginning of trigonometry as a subject separate from astronomy. In his treatise *De triangulis omnimodis* (about 1464) we find among the many formulas, the formula for the area of a triangle, $A = \frac{1}{2} ab \sin C$.

CUMULATIVE REVIEW

Chapters XIII, XIV, and XV

1. Which of these statements are true? Which are false?

(*a*) $\log_3 \frac{1}{9} = \log_{10} 100$.

(*b*) If $b^x = a$, then $x = \dfrac{\log a}{\log b}$.

(*c*) If the angle of elevation of the top of a tower as seen from a point m feet horizontally from its base is $27° \, 18'$, the height of the tower is less than m feet.

(*d*) In the right triangle in which $a = 7$, $b = 24$, and $c = 25$, $\log \cos B > \log \cos A$.

(*e*) The number of different triangles that may be formed when $a = 40$, $b = 50$, and $A = 30°$ is two.

(*f*) In triangle ABC, if b and c are constant in length, then as angle A increases from $0°$ to $90°$ the area of triangle ABC increases.

(*g*) If sides a and b of triangle ABC include an angle less than $30°$, then the area of triangle ABC is less than $\frac{1}{4}\,ab$.

(*h*) The area of triangle ABC is always determined if the measurements for a, b, and A are given.

(*i*) If the area of a triangle is 14 and two of its sides are 5 and 7, then the sine of the included angle is .8000.

(*j*) The formula for the area of a triangle $K = \dfrac{a^2 \sin B \sin C}{2 \sin(B + C)}$ may be put into the form $K = \frac{1}{2}\,a^2 \sin B \sin C \csc A$.

2. Complete each of the following statements:

(*a*) If A is an acute angle such that $\log \cos A = 9.3844 - 10$, then $A = ?$ correct to the nearest minute.

(*b*) Antilog $\bar{2}.5135 = ?$

(*c*) In the oblique triangle ABC, if $a = 3$, $b = 4$, and $\cos C = \frac{3}{8}$, then $c = ?$

(*d*) If two sides of a triangle are 4 and 6 and include an angle of $60°$, the third side is ? . (Give answer in radical form.)

(*e*) In triangle ABC, if $\sin A = \frac{1}{5}$, $\sin B = \frac{3}{5}$, and $a = 100$, then $b = ?$

(*f*) In triangle ABC, if $b = 100$, $c = 200$, and $A = 40° 15'$, then the area of triangle $ABC = ?$

(*g*) If the base of an isosceles triangle is 8 inches and each of the base angles is 61°, then the area of the triangle is ? .

(*h*) If the area of triangle $ABC = 36$ and $B = C = 45°$, then $a = ?$

3. Solve for x: $(5.3)^{x+1} = 59.7$.

4. Solve the equation $10 \sin^2 x + 3 \sin x - 1 = 0$ for all values of x between 0° and 360°. Determine the approximate values of x correct to the nearest minute.

5. An observer at station A notes that the angle of elevation of the top of a mountain is 19° 18′. He travels in a horizontal plane 2200 feet directly toward the mountain to station B where he finds that the angle of elevation is 34° 43′. How far is B from the top of the mountain?

6. Two sides of a triangular plot of ground are 155 feet and 96 feet respectively and the angle opposite the greater side is 43° 18′. Find the third side correct to the nearest foot.

7. Prove that two triangles are equal in area if two sides of the first triangle are equal respectively to two sides of the second and the angles included between these sides are supplementary.

8. (*a*) Find the area of a rhombus if one of its sides is 11.7 in. and one of its angles is 41° 14′.

(*b*) Find the lengths of the diagonals of the rhombus in (*a*).

9. Find the area of parallelogram $ABCD$ in which $AB = 10.3$ inches, $\angle ABD = 22° 15'$ and $\angle DBC = 18° 45'$.

10. The area of a circle is 64π square inches. Find the perimeter and area of the regular circumscribed pentagon.

11. A triangular plot of ground is pictured on a map in which 1 inch represents 25 feet. Find the actual area of the plot if its sides on the map measure 7.2 inches, 5.3 inches, and 4.7 inches.

CHAPTER XVI. ARITHMETIC AND GEOMETRIC PROGRESSIONS

The new mathematics is a sort of supplement to language, affording a means of thought about form and quantity and a means of expression, more exact, compact, and ready, than ordinary language. — H. G. Wells.

Series or Progressions

Problem. Each member of a Christmas Club deposits in a bank 2 cents the first week and each succeeding week increases his deposit 2 cents. What is the total money deposited by each member after the 50th deposit?

This problem may be solved by finding the deposit each week for 50 weeks and then adding the weekly deposits. It is possible, however, to solve such problems as this by a much simpler method as the following presentation will illustrate.

Let us examine each of the following sequences of numbers. Write the next number in each sequence.

1. $1, 2, 3, 4, 5, \cdots$.

2. $2, 5, 8, 11, 14, \cdots$.

3. $16, 12, 8, 4, 0, \cdots$.

4. $1, 2, 4, 8, 16, \cdots$.

5. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$.

6. $2, -\frac{2}{3}, \frac{2}{9}, -\frac{2}{27}, \frac{2}{81}, \cdots$.

Careful examination of these sequences of numbers will show that each follows a definite law. This definite law, however, is different in each sequence. For instance, each number in the first sequence may be obtained from the number immediately preceding it by the addition of 1. Similarly each number may be found from the preceding one in:

Sequence 2 by the addition of 3.

Sequence 3 by the addition of -4.

Sequence 4 by the multiplication by 2.

Sequence 5 by the multiplication by $\frac{1}{2}$.

Sequence 6 by the multiplication by $-\frac{1}{3}$.

A sequence of numbers, in which the numbers (*terms*) follow a definite order according to some definite uniform relation or law, is called a *series* or a *progression*.

ORAL EXERCISES

By discovering its law, continue each of the following series to four additional terms.

1. $2, 4, 6, 8, \cdots$.
2. $4, 2, 1, \frac{1}{2}, \cdots$.
3. $24, 21, 18, 15, \cdots$.
4. $7, 5, 3, 1, \cdots$.
5. $5, 0, -5, -10, \cdots$.
6. $a, -\frac{a}{2}, \frac{a}{4}, -\frac{a}{8}, \cdots$.
7. $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, \cdots$.
8. $\sqrt{2}, 2, 2\sqrt{2}, \cdots$.
9. $x, x+y, x+2y, \cdots$.
10. $x, -x^2, x^3, -x^4, \cdots$.
11. $\tan x, 2\tan x, 3\tan x, \cdots$.
12. $\cos x, 0, -\cos x, \cdots$.
13. $a, a+d, a+2d, \cdots$.
14. $a, ar, ar^2, \cdots$.
15. $p, p+2q, p+4q, \cdots$.
16. $b, bc^2, bc^4, \cdots$.
17. $t, -rt, r^2t, -r^3t, \cdots$.
18. $w, 2wt, 4wt^2, \cdots$.

The Arithmetic Series or Progression

An *arithmetic progression* is a series in which each term after the first is obtained by adding the same quantity to the term before it. The fixed or constant quantity added to each term of the series to obtain the next is called the *common difference*, because it may be found by subtracting any term from the one which immediately follows it. Note that the common difference may be positive or negative. When it is positive, the series is called an *increasing series*, and when negative, it is called a *decreasing series*.

Thus $2, 4, 6, 8, \cdots$ is an increasing arithmetic progression, and $24, 21, 18, 15, \cdots$ is a decreasing arithmetic progression.

Formula for the *n*th or last term. If the first term of an arithmetic progression is represented by a and the common difference by d, the series may be represented by:

$$a, (a + d), (a + 2d), (a + 3d), \cdots.$$

Let us arrange the terms in tabular form and discover any existing relationships. By inspection of these terms we observe:

Number of Term	The Term Is
1st	a (or $a + 0d$)
2nd	$a + (1)d$
3rd	$a + 2d$
4th	$a + 3d$
5th	$a + 4d$
$\cdots$	$\cdots$
nth	$a + (n - 1)d$

1. Each term contains a.

2. Each term contains some number (coefficient) times d.

3. The coefficient of d in each term is 1 less than the number of that term.

4. Since the last term is the nth term, the coefficient of d in that term must be $(n - 1)$. Hence, if l represents the nth or last term, the general formula for the nth term may be written as:

$$\boldsymbol{l = a + (n - 1)d.}$$

This formula enables us to find any one of the four letters l, a, n, or d, when the other three are known.

Illustrative examples.

Example 1. Find the 13th term of the series 12, 23, 34, 45, $\cdots$.

Solution

In this series $a = 12$, $d = 11$, $n = 13$.

Substituting in the formula

$$l = a + (n - 1)d,$$

$$l = 12 + (13 - 1)11 = 12 + 132 = 144.$$

Example 2. If the 9th term of an arithmetic progression is 12, and the 17th term is 60, find the first term and the common difference, and write the progression through the 5th term.

Solution

Since the 9th term can be written as $a + 8d$, and the 17th term as $a + 16d$, we then have

$$\begin{aligned} a + 8d &= 12 \qquad (1) \\ a + 16d &= 60 \qquad (2) \end{aligned}$$

Subtracting $\quad -8d = -48$

$$d = 6.$$

Substituting $d = 6$ in (1) $\quad a + 48 = 12.$

$$\therefore a = -36.$$

Hence the series is $-36, -30, -24, -18, -12, \cdots$.

EXERCISES

In each of the following find the term required:

1. The 20th term of $2, 4, 6, 8, \cdots$.

2. The 30th term of $1, 3, 5, 7, \cdots$.

3. The 14th term of $-7, -4, -1, \cdots$.

4. The 12th term of $3, \frac{3}{2}, 0, \cdots$.

5. The 8th term of $1\frac{3}{4}, 1, \frac{1}{4}, \cdots$.

6. The 9th term of $2, 2\frac{1}{4}, 2\frac{1}{2}, \cdots$.

7. The 16th term of $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, \cdots$.

8. The 20th term of $1.03, 1.06, 1.09, \cdots$.

9. The 8th term of $\tan x, 3\tan x, 5\tan x, \cdots$.

10. The 6th term of $5\sin x, 3\sin x, \sin x, \cdots$.

11. The 7th term of $-7\cos x, -4\cos x, -\cos x, \cdots$.

12. The 8th term of $\tan x, \frac{5}{4}\tan x, \frac{3}{2}\tan x, \cdots$.

13. The 7th term of an arithmetic progression is 12 and the 12th term is 7. Find the common difference and write the series through 5 terms.

14. The 3rd term of an arithmetic progression is 16 and the 8th term is 1. Write the series through 3 terms.

15. The 4th term of an arithmetic progression is -7 and the 9th term is -37. Write the series through 4 terms.

16. Which term of the series $3, 7, 11, \cdots$ is 47?

Arithmetic means. If the numbers 6 and 8 are written in order between 4 and 10, they form the arithmetic progression 4, 6, 8, 10. In this progression 6 and 8 are called the arithmetic means between 4 and 10.

In general in an arithmetic series, the terms between any two terms that are not consecutive are called *arithmetic means.*

Any number of arithmetic means may be inserted between two given numbers, and the following illustrative example will show how this can be done.

Illustrative example. Insert 5 arithmetic means between 2 and 26.

Solution

Here $a = 2$, $l = 26$, and since the 5 arithmetic means with the first and last terms make a series of 7 terms, then $n = 7$.

Substituting these values in

$$\begin{aligned} l &= a + (n - 1)d, \\ 26 &= 2 + (7 - 1)d. \\ 26 &= 2 + 6\,d. \\ d &= 4. \end{aligned}$$

Hence the series is 2, 6, 10, 14, 18, 22, 26 and the means are 6, 10, 14, 18, 22

One arithmetic mean inserted between two given numbers is called *the arithmetic mean.* To insert only one arithmetic mean between two given numbers is much easier than to insert several, as the following simple formula shows:

If a and b are the two given numbers and M the required arithmetic mean, the series is a, M, b.

Now $\qquad M - a =$ the common difference

and $\qquad b - M =$ the common difference.

But since the common difference is a constant quantity in an arithmetic progression, $M - a = b - M$. Why?

$$\therefore M = \frac{a + b}{2}.$$

Illustrative example. Find the arithmetic mean between 10 and 15.

Solution

Here $a = 10, \quad b = 15.$

Substituting in $M = \dfrac{a+b}{2},$

$$M = \frac{10 + 15}{2} = \frac{25}{2} = 12\tfrac{1}{2}.$$

EXERCISES

1. Insert 8 arithmetic means between 4 and 28.
2. Insert 7 arithmetic means between 3 and -1.
3. Insert 4 arithmetic means between 2 and 18.
4. Insert 4 arithmetic means between $\frac{3}{2}$ and -11.
5. Insert 4 arithmetic means between -6 and 14.
6. Insert 2 arithmetic means between -1 and 0.

Find the arithmetic mean between:

7. 3 and 7.
8. 5 and 4.
9. x and y.
10. -7 and -5.
11. 15 and $-3\frac{1}{4}$.
12. $\dfrac{m}{2}$ and $\dfrac{n}{2}$.
13. $a + b$ and $a - b$.
14. $6 \sin x$ and $4 \sin x$.
15. $\sin x + \cos x$ and $\sin x - \cos x$.
16. $\tan x$ and $\cot x$.

The formula for the sum of the terms. If s represents the sum of the terms of an arithmetic series,

$$s = a + (a + d) + (a + 2\,d) + \cdots + (l - 2\,d) + (l - d) + l.$$

Reversing

$$s = l + (l - d) + (l - 2\,d) + \cdots + (a + 2\,d) + (a + d) + a.$$

Adding

$$2\,s = (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) + (a+l).$$

$\therefore\ 2\,s = n(a + l)$, for there is an $(a + l)$ for each term of the series.

$$\therefore\ s = \frac{n(a + l)}{2}. \qquad (1)$$

When, however, d is given instead of l, we may derive a more convenient formula for the sum by substituting $a + (n - 1)d$ for l in (1), and get

$$s = \frac{n}{2}[a + a + (n - 1)d]$$

or

$$s = \frac{n}{2}[2a + (n - 1)d].$$

Illustrative examples.

Example 1. Each member of a Christmas Club deposits in a bank 2 cents the first week and each succeeding week increases his deposit 2 cents. What is the total money deposited by each member after the 50th deposit?

Solution

Here the series is 2, 4, 6 $\cdots$ to 50 terms. Therefore

$$a = 2, \quad d = 2, \quad n = 50.$$

Substituting in $\quad s = \frac{n}{2}[2a + (n - 1)d],$

$$s = \tfrac{50}{2}[4 + (50 - 1)2]$$
$$= 2550 \text{ cents or } \$25.50.$$

Example 2. Find the number of terms of the series 5, 10, 15, 20, $\cdots$ that must be taken to give the sum 1050.

Solution

In this series, $\quad a = 5, \; d = 5, \; s = 1050.$

Substituting in $\quad s = \frac{n}{2}[2a + (n - 1)d],$

$$1050 = \frac{n}{2}[10 + (n - 1)5].$$
$$2100 = n(10 + 5n - 5).$$
$$2100 = 5n^2 + 5n.$$
$$n^2 + n - 420 = 0.$$
$$(n + 21)(n - 20) = 0.$$
$$\therefore n = -21; \; n = 20.$$

As a negative or a fractional number of terms is impossible, the answer is 20.

EXERCISES

1. Find s when $a = 3$, $l = 30$, $n = 10$.
2. Find s when $a = -6$, $l = 72$, $n = 12$.
3. Find s when $a = \frac{1}{5}$, $l = \frac{15}{16}$, $n = 14$.
4. Find s when $a = \sqrt{3}$, $l = 19\sqrt{3}$, $n = 10$.

Find the sum of the first:

5. 15 terms of $3, 7, 11, \cdots$.
6. 7 terms of $\frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \cdots$.
7. 12 terms of $-4, -2, 0, \cdots$.
8. 8 terms of $\sin x, 2 \sin x, 3 \sin x, \cdots$.
9. 6 terms of $5 \sin x, 3 \sin x, \sin x, \cdots$.
10. 8 terms of $-7 \cos x, -3 \cos x, \cos x, \cdots$.
11. 18 terms of $2 \cos x, \frac{3}{2} \cos x, \cos x, \cdots$.
12. 25 integral numbers.
13. 25 positive odd numbers.
14. 25 negative even numbers.

15. A man was paid for boring a well 100 yards deep as follows: \$3 for the first yard, \$3.25 for the second, \$3.50 for the third, and so on. How much was he paid?

16. A clerk receives \$60 a month for his first year, and an annual increase of \$100 for the next ten years. Find his salary for the 11th year and his total salary for the 11 years.

17. A body falling freely from rest falls 16 feet the first second, 48 feet the second second, 80 feet the third second, and so on. Find the distance a body would fall during the seventh second, and the total distance it would fall during the first 7 seconds.

18. Twelve potatoes are placed in line at distances 6, 12, 18, $\cdots$ feet from a basket. A player, starting from the basket, must pick up the potatoes one at a time and carry each back to the basket. How far must he run to complete the potato race?

The Geometric Series or Progression

A *geometric progression* is a series in which each term after the first is obtained by multiplying the term before it by the same quantity. The fixed or constant quantity by which each term of the series is multiplied to obtain the next is called the *common ratio,* because it may be found by dividing any term by the one which immediately precedes it. The common ratio may be positive or negative. When it is positive, the series may be increasing or decreasing, depending upon whether this common ratio is greater than or less than 1.

Thus 2, 4, 8, 16, $\cdots$ is an increasing geometric series (common ratio is $+2$); while 10, 5, $2\frac{1}{2}$, $1\frac{1}{4}$, $\cdots$ is a decreasing geometric series (common ratio is $+\frac{1}{2}$).

When the common ratio is negative, the series is an *oscillating series,* since it alternately increases and decreases. Thus 2, -4, 8, -16, $\cdots$ is an oscillating series (common ratio -2).

Formula for the *n*th or last term. If the first term of a geometric progression is represented by a and the common ratio by r, the series can be represented by:

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots.$$

Let us arrange the terms in tabular form and discover any existing relationships. By inspection of these terms we observe:

Number of Term	The Term Is
1st	a (or ar^0)
2nd	$ar^{(1)}$
3rd	ar^2
4th	ar^3
5th	ar^4
$\cdots$	$\cdots$
nth	ar^{n-1}

1. Each term contains a.

2. Each term contains r with some exponent (first term contains r^0).

3. The exponent of r is one less than the number of the term.

4. Since the last term is the nth term, the exponent of r in that term must be $(n-1)$. Hence, if l represents the nth or last term, the general formula for that term may be written as

$$\boldsymbol{l = ar^{n-1}}.$$

This formula enables us to find any one of the four letters, l, a, r, or n, when the other three are known.

Illustrative examples.

Example 1. Find the 10th term of the series 2, 4, 8, 16, $\cdots$.

Solution

In this series $\quad a = 2, \quad r = 2, \quad n = 10.$

Substituting in the formula $l = ar^{n-1}$,

$$\begin{aligned} l &= 2(2)^9 \\ &= 2 \times 512 \\ &= 1024. \end{aligned}$$

Example 2. If the 7th term of a geometric progression is 192, and the 12th term is 6144, find the first term and the common ratio, and write the progression through the 5th term.

Solution

Since the 7th term can be written ar^6 and the 12th term as ar^{11}, we have

$$ar^6 = 192. \tag{1}$$

$$ar^{11} = 6144. \tag{2}$$

Dividing equation (2) by (1), we have

$$\begin{aligned} r^5 &= 32. \\ \therefore r &= 2. \end{aligned}$$

Substituting $r = 2$ in equation (1)

$$\begin{aligned} a(2)^6 &= 192. \\ \therefore a &= 3. \end{aligned}$$

Hence the series is 3, 6, 12, 24, 48, $\cdots$.

EXERCISES

Find:

1. The 6th term of 2, 4, 8, 16, $\cdots$.
2. The 8th term of 6, $-$ 12, 24, $\cdots$.
3. The 7th term of 16, $-$ 4, 1, $\cdots$.

4. The 8th term of 27, 9, 3, $\cdots$.
5. The 10th term of 1, x^2, x^4, $\cdots$.
6. The 10th term of $\sqrt{3}$, 3, $3\sqrt{3}$, $\cdots$.
7. The 6th term of .5, .05, .005, $\cdots$.
8. The 20th term of 1, 1.05, $(1.05)^2$, $\cdots$.
9. The 8th term of $\tan x$, $\tan^2 x$, $\tan^3 x$, $\cdots$.
10. The 7th term of $\cos x$, $-3\cos^2 x$, $9\cos^3 x$, $\cdots$.
11. The 6th term of $\sin x$, $\sin x \cos x$, $\sin x \cos^2 x$, $\cdots$.
12. The 8th term of $\sin x$, $\tan x$, $\sin x \sec^2 x$, $\cdots$.
13. The 3rd term of a geometric progression is 20 and the 8th term is 640. Find the common ratio, and write the series through 5 terms.
14. The 4th term of a geometric progression is 54 and the 8th term is 4374. Write the two series through 3 terms.
15. Which term of the series 81, 27, 9, $\cdots$ is $\frac{1}{243}$?

Geometric means. If the numbers 4 and 8 are written in order between 2 and 16, they form the geometric progression 2, 4, 8, 16. In this progression 4 and 8 are called geometric means between 2 and 16. In general, in a geometric series, the terms between any two terms that are not consecutive, are called *geometric means.*

Illustrative example. Insert 3 geometric means between 3 and 243.

Solution

Here $a = 3$, $l = 243$, and since the 3 means with the first and last terms make a series of 5 terms, then $n = 5$.

Substituting these values in $l = ar^{n-1}$,

$$243 = 3(r)^4.$$
$$81 = r^4.$$
$$r = \pm 3.$$

If $r = +3$, the series is 3, 9, 27, 81, 243.

If $r = -3$, the series is 3, -9, 27, -81, 243.

One geometric mean inserted between two given numbers is called *the geometric mean.* To insert only one geometric mean between two given numbers is much easier than to insert several, as the following simple formula shows:

If a and b are the two given numbers and M the required geometric mean, the series is a, M, b.

Now $$\frac{M}{a} = \text{the common ratio}$$

and $$\frac{b}{M} = \text{the common ratio.}$$

But since the common ratio is a constant quantity in a progression, $$\frac{M}{a} = \frac{b}{M}.$$

$$\therefore M^2 = ab$$

or $$\boldsymbol{M = \pm\sqrt{ab}.}$$

Something to think about. Prove that the geometric mean between two quantities is the mean proportional between them.

Illustrative example. Find the geometric mean between 2 and 32.

Solution

Here $a = 2, \quad b = 32.$

Substituting in $M = \pm\sqrt{ab},$

$$M = \pm\sqrt{2 \times 32} = \pm 8.$$

EXERCISES

1. Insert 4 geometric means between 5 and 160.
2. Insert 5 geometric means between $\frac{3}{4}$ and 48.
3. Insert 3 geometric means between 12 and $\frac{3}{4}$.
4. Insert 5 geometric means between 9 and 576.
5. Insert 4 geometric means between $-\frac{1}{2}$ and 16.
6. Insert 4 geometric means between $\sqrt{3}$ and $4\sqrt{6}$.

Find the geometric mean between:

7. 5 and 20. **9.** 2 and 14. **11.** -5 and -10.

8. $\frac{1}{2}$ and 32. **10.** $\frac{1}{8}$ and $\frac{1}{2}$. **12.** $\frac{m}{n}$ and $\frac{n}{m}$.

13. $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$.

14. $\frac{x+y}{x-y}$ and $\frac{1}{x^2-y^2}$.

15. $9 \sin x$ and $4 \sin x$.

16. $\tan x$ and $\sin x \cos x$.

17. $\cos^2 x$ and $\tan^2 x$.

18. $\cot x$ and $\sin^3 x \sec x$.

The formula for the sum of the terms. If s represents the sum of the terms of a geometric series,

$$s = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}.$$

Multiplying by r

$$rs = \quad ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n.$$

Subtracting

$$s - rs = a - ar^n.$$

Factoring

$$s(1 - r) = a - ar^n.$$

$$\therefore\ \mathbf{s = \frac{a - ar^n}{1 - r}}. \qquad (1)$$

If r is greater than one, negative quantities can be avoided by writing equation (1) thus:

$$\mathbf{s = \frac{ar^n - a}{r - 1}}.$$

These formulas are used to find the value of s when the values of a, r, and n are known. However, the value of l is frequently given instead of the value of n. To solve such cases additional formulas may be derived for the sum.

Since $l = ar^{n-1}$, if we multiply both members by r, then $rl = ar^n$.

Substituting rl for ar^n in the formulas

$$s = \frac{a - ar^n}{1 - r} \quad \text{and} \quad s = \frac{ar^n - a}{r - 1},$$

we have $$s = \frac{a - rl}{1 - r} \quad \text{or} \quad s = \frac{rl - a}{r - 1}.$$

Illustrative examples.

Example 1. Find the sum of the first 5 terms of the series $3, -9, 27, \cdots$.

Solution

Here $\quad a = 3, \quad r = -3, \quad n = 5.$

Substituting in $\quad s = \dfrac{a - ar^n}{1 - r},$

$$s = \frac{3 - 3(-3)^5}{1 - (-3)} = \frac{3 + 729}{4} = 183.$$

Example 2. Find a series of numbers in geometric progression such that the first is -40, the last $\frac{5}{4}$, and the sum of the series is $-\frac{105}{4}$.

Solution

Here $\quad a = -40,\ l = \frac{5}{4},\ s = -\frac{105}{4}.$

Substituting in $\quad s = \dfrac{rl - a}{r - 1},$

$$-\frac{105}{4} = \frac{r(\frac{5}{4}) - (-40)}{r - 1}.$$

$$-105(r - 1) = 4\left(\frac{5r}{4} + 40\right).$$

$$r = -\tfrac{1}{2}.$$

The series is $-40, 20, -10, 5, -\frac{5}{2}, \frac{5}{4}$.

Notice that n can be found if required, before the series is obtained, by substituting the values of l, a, and r in the formula

$$l = ar^{n-1}.$$

EXERCISES

Find the sum of the first:

1. 5 terms of $2, 6, 18, \cdots$.

2. 7 terms of $3, -6, 12, \cdots$.

3. 6 terms of $-4, 12, -36, \cdots$.

4. 5 terms of $3, -\frac{1}{3}, \frac{1}{27}, \cdots$.

5. 6 terms of $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \cdots$.

6. 6 terms of $3\frac{3}{8}, 2\frac{1}{4}, 1\frac{1}{2}, \cdots$.

7. 8 terms of $1, -\frac{1}{2}, \frac{1}{4}, \cdots$.

8. 5 terms of $\tan x, \tan^2 x, \tan^3 x, \cdots$.

9. 8 terms of $5 \sin x, -10 \sin^2 x, 20 \sin^3 x, \cdots$.

10. 5 terms of $\sin x, \tan x, \sin x \sec^2 x, \cdots$.

11. The first term of a geometric progression is 1, the last term is $\frac{1}{16}$, and the sum $\frac{11}{16}$. Find the ratio and write the series through 4 terms.

12. Find the first and last terms of a geometric series of 6 terms whose sum is $3\frac{15}{16}$ and whose common ratio is $\frac{1}{2}$.

13. If a man worked 10 days and received 2 cents the first day, 4 cents the second, 8 cents the third, and so on, how much will he receive the tenth day, and how much in all?

14. A chain letter is written, each person receiving the letter rewriting it and sending it to two others. If the person who starts the chain sends out two letters, how many will have been written after ten sets of letters have been sent?

15. The first stroke of an air pump exhausts one-tenth of the air in a bell jar and each succeeding stroke exhausts one-tenth of the remaining air. How much of the air originally in the receiver is left after 5 strokes?

16. An automobile salesman tells me that my car each year is worth 45% of the value it had the preceding year. If my car cost $1800, what is its value at the end of 3 years?

17. The population of a certain city is 400,000, and it normally increases 50% every 10 years. What will the population be in 50 years if it continues to grow at this rate?

18. One dollar at 6% interest compounded annually amounts in a year to $1.06. Similarly if any sum of money is placed at interest at the rate of 6% compounded annually, the amount at the end of the first year can be obtained by multiplying the principal by 1.06; at the end of the second year by multiplying this amount again by 1.06 or by multiplying the original sum by $(1.06)^2$. By what would the original sum be multiplied at the end of the third year? the fourth year? the tenth year? the nth year?

19. If money is placed at 4% interest compounded annually, by what should the principal be multiplied to obtain the amount at the end of 1 yr.? 2 yr.? 10 yr.?

20. If interest is compounded annually, by what should a principal of $1 be multiplied to obtain the amount at the end of 2 yr.? 3 yr.? 10 yr.? n yr.?

21. Make a formula for the amount of P dollars at rate r, interest compounded annually for n years.

22. What is the amount of P dollars at rate r, interest compounded annually for 5 yr.? 4 yr.? 3 yr.? 2 yr.? 1 yr.? Do they form a geometric progression? What is the ratio? What is the sum?

23. The first 3 terms of a geometric progression are P, $P(1+r)$, $P(1+r)^2$.

(*a*) Write the 4th term; the 10th term; the nth term.

(*b*) Find the sum of the first 7 terms; 12 terms; n terms.

24. At the beginning of each year Mr. Walch places $1000 at 5% compound interest. How much will he have saved just after making the deposit at the beginning of the 5th year?

25. At the beginning of each year Mr. George pays $100 premium to an insurance company. If money is worth 3%, interest compounded annually, how much should the company pay him at the end of 20 yr.? (Use logarithms.)

26. A sum of money earns 4% interest annually, compounded semiannually. By what number should the principal be multiplied to obtain the amount at the end of the first half year? of the first year? of the second year? of the 9th year?

The Infinite Geometric Progression

Problem. A boy planned to walk according to the following schedule: 2 miles the first hour, 1 mile the second hour, $\frac{1}{2}$ the third hour, and so on. How far would the boy walk if he continued this schedule indefinitely?

Discussion

Evidently the distances traveled each hour form the series 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\cdots$, which is a decreasing geometric progression in which $a = 2$ and $r = \frac{1}{2}$. His progress may be represented graphically thus:

The total distance he has traveled at the end of each successive hour may be arranged as in the following table:

During	Distance	At End of	Total Distance
1st hour	2	1st hour	2
2nd hour	1	2nd hour	3
3rd hour	$\frac{1}{2}$	3rd hour	$3\frac{1}{2}$
4th hour	$\frac{1}{4}$	4th hour	$3\frac{3}{4}$
5th hour	$\frac{1}{8}$	5th hour	$3\frac{7}{8}$
6th hour	$\frac{1}{16}$	6th hour	$3\frac{15}{16}$
7th hour	$\frac{1}{32}$	7th hour	$3\frac{31}{32}$
8th hour	$\frac{1}{64}$	8th hour	$3\frac{63}{64}$
9th hour	$\frac{1}{128}$	9th hour	$3\frac{127}{128}$
10th hour	$\frac{1}{256}$	10th hour	$3\frac{255}{256}$
11th hour	$\frac{1}{512}$	11th hour	$3\frac{511}{512}$
12th hour	$\frac{1}{1024}$	12th hour	$3\frac{1023}{1024}$

We can observe from the table that:

1. Distance traveled each hour is one-half the distance traveled the preceding hour.

2. As the number of hours increases, the distance traveled during each hour decreases, becoming a smaller and smaller fraction of a mile and approaching nearer and nearer to zero.

3. As the number of hours increases the sum of the distances traveled increases, approaching nearer and nearer to 4. This sum, however, will never reach 4 because the numerator of the fractional part of the distance is always 1 less than the denominator. But if we continue sufficiently far, the difference between the total distance and 4 becomes so small that it is practically negligible. Hence, we say that the total distance approaches the value 4 as its *limit*.

Thus, since in a decreasing geometric series the common ratio is less than one, it is possible to find a number beyond which the sum cannot pass, and to which we can come closer and closer by increasing the number of terms. When the number of terms, n, becomes larger and larger, we say *n increases indefinitely* or *n approaches infinity*, and the limit to which the sum then approaches is called the *sum of a geometric progression to infinity.* The symbol for this limiting value of the sum is s_∞.

Formula for the "Sum to Infinity." As you know, the formula for the sum of a geometric progression, when r is less than 1, is $s = \frac{a - ar^n}{1 - r}$. This formula may be written as the difference between two fractions:

$$s = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

Let us examine the second fraction and observe what happens when n increases indefinitely. Since r is less than 1, the successive powers of r become smaller and smaller. Why?

Thus if $r = \frac{1}{2}$ then $r^2 = \frac{1}{4}$, $r^3 = \frac{1}{8}$, $r^5 = \frac{1}{32}$, $\cdots r^{11} = \frac{1}{2048} \cdots$. Therefore, r^n approaches zero as n increases indefinitely, and no matter what the value of a may be, the value of the numerator, ar^n, still approaches zero. Further, since the denominator does not change, the entire fraction $\frac{ar^n}{1-r}$ approaches zero, as n becomes infinitely large. Therefore, the right-hand member of the formula for the sum, $\frac{a}{1-r} - \frac{ar^n}{1-r}$, approaches $\frac{a}{1-r}$ as n becomes infinitely large. Consequently, we may find the limiting value of the sum of a decreasing geometric progression by using the formula

$$\boldsymbol{s_\infty = \frac{a}{1-r}}.$$

Now let us return to our problem, in which it is required to find the sum to infinity of the decreasing geometric progression $2, 1, \frac{1}{2}, \frac{1}{4}, \cdots$ in which $a = 2$ and $r = \frac{1}{2}$.

Solution

Substituting in the formula $s_\infty = \frac{a}{1-r}$,

$$s_\infty = \frac{2}{1-\frac{1}{2}} = \frac{2}{\frac{1}{2}} = 4.$$

Illustrative example. Find the sum of the infinite geometric progression: $3, -1, \frac{1}{3}, \cdots$.

Solution

In this series $a = 3, \quad r = -\frac{1}{3}$.

Substituting in the formula $s_\infty = \frac{a}{1-r}$,

$$s_\infty = \frac{3}{1-(-\frac{1}{3})} = \frac{3}{1+\frac{1}{3}} = \frac{9}{4} = 2\frac{1}{4}.$$

EXERCISES

Find the sum of each of the following infinite geometric progressions:

1. $1, \frac{1}{2}, \frac{1}{4}, \cdots$.

2. $6, 3, \frac{3}{2}, \cdots$.

3. $15, 5, \frac{5}{3}, \cdots$.

4. $1, \frac{3}{4}, \frac{9}{16}, \cdots$

5. $5, -\frac{1}{2}, \frac{1}{20}, \cdots$.

6. $\frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, \cdots$.

7. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \cdots$.

8. $3, \frac{3}{2}, \frac{3}{4}, \cdots$.

9. $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \cdots$.

10. $3\frac{3}{8}, 2\frac{1}{4}, 1\frac{1}{2}, \cdots$.

11. A ball falls from a height of 50 feet and at every fall rebounds half the distance. What is the total distance passed through before the ball comes to rest?

12. What distance will an elastic ball travel before coming to rest, if it is dropped from a height of 15 feet, and if after each fall it rebounds two-thirds of the height from which it last fell?

13. A sled after reaching the foot of a toboggan slide goes 25 feet the first second. Each second thereafter it goes $\frac{3}{5}$ as far as in the previous second. What is the greatest distance the sled can go?

14. Given a square whose side is 24. The middle points of its sides are joined successively by straight lines forming a second square inscribed in the first. In the same manner, a third square is inscribed in the second, a fourth in the third, and so on indefinitely. Find the sum of the perimeters of all the squares.

An interesting application of the "Sum to Infinity." If a common fraction is changed into decimal form, it is found that the resulting decimal either "comes out even," or a certain sequence of integers is repeated indefinitely.

Thus: 1. $\frac{1}{2} = .5$.
2. $\frac{3}{8} = .375$.
3. $\frac{3}{100} = .03$.
4. $\frac{3}{4} = .75$.
5. $\frac{9}{16} = .5625$.
6. $\frac{1}{3} = .3333333 \cdots$.
7. $\frac{2}{7} = .285714285714 \cdots$.
8. $\frac{17}{18} = .9444444 \cdots$.
9. $\frac{41}{110} = .3727272 \cdots$.
10. $\frac{2911}{5550} = .52450450 \cdots$.

Decimals which have a sequence of integers repeated indefinitely are called *repeating decimals* or *recurring decimals.* In arithmetic, you learned how to change a common fraction into a decimal, but never have you been able to change a repeating decimal back into an equivalent common fraction. The following illustrations will show you how this can be done by the use of the infinite geometric progression.

Illustrative examples.

Example 1. Change the repeating decimal .5555 $\cdots$ to an equivalent fraction.

Solution

The decimal .5555 $\cdots$ may be written as $.5 + .05 + .005 + .0005 + \cdots$. This series, you will observe, is a geometric progression in which $a = .5$, $r = .1$, and n is infinitely large.

Substituting in $s_\infty = \dfrac{a}{1 - r}$,

$$s_\infty = \frac{.5}{1 - .1} = \frac{.5}{.9} = \frac{5}{9}.$$

Example 2. Change .545454 $\cdots$ to an equivalent fraction.

Solution

The decimal .545454 $\cdots$ may be written as $.54 + .0054 + .000054 + \cdots$.

In this series $a = .54$, $r = .01$, and n is infinitely large.

Substituting in $s_\infty = \dfrac{a}{1 - r}$,

$$s_\infty = \frac{.54}{1 - .01} = \frac{.54}{.99} = \frac{6}{11}.$$

Example 3. Change .25555 $\cdots$ to an equivalent fraction.

Solution

The decimal .25555 $\cdots$ may be written as $.2 + [.05 + .005 + .0005 + \cdots]$.

Here the series consists of two parts:

(a) the .2 and (b) the series beginning with .05.

First sum the series (*b*) in which $a = .05, \quad r = .1$.

Substituting in $s_\infty = \dfrac{a}{1 - r}$,

$$s_\infty = \frac{.05}{1 - .1} = \frac{.05}{.90} = \frac{1}{18}.$$

Now adding (*a*) and (*b*) above, we have $.2 + \frac{1}{18}$, *i.e.*,

$$.25555 \cdots = \tfrac{1}{5} + \tfrac{1}{18} = \tfrac{23}{90}.$$

EXERCISES

Change the following repeating decimals to equivalent fractions:

1. $.444444 \cdots$.
2. $.666666 \cdots$.
3. $.777777 \cdots$.
4. $.232323 \cdots$.
5. $.343434 \cdots$.
6. $.565656 \cdots$.
7. $.787878 \cdots$.
8. $.818181 \cdots$.
9. $.454545 \cdots$.
10. $.727272 \cdots$.
11. $.328328 \cdots$.
12. $.189189 \cdots$.
13. $.645645 \cdots$.
14. $.45555 \cdots$.
15. $.37777 \cdots$.
16. $.42333 \cdots$.
17. $.23737 \cdots$.
18. $.56253253 \cdots$.

EXERCISES

Group A

1. Find the sum to 12 terms of $4, 9, 14, \cdots$.
2. Find the sum to 12 terms of $x + y, x, x - y, \cdots$.
3. Find the sum of the progression $3, 6, 12, \cdots$ to 9 terms.
4. Insert 8 arithmetic means between 28 and 4.
5. Find the sum of the infinite geometric series $4, \frac{8}{3}, \frac{16}{9}, \cdots$.
6. Find the geometric mean between $x^2 - y^2$ and $\dfrac{x + y}{x - y}$.
7. Express $.53131 \cdots$ as a common fraction.

8. The first term of an arithmetic progression is -7 and the common difference is 4. Find the sum of the first 23 terms.

9. Find the sum of the first 8 terms in a geometric progression if the third term is 12 and the sixth term is 96.

10. Express as a power of $\frac{1}{2}$ the 20th term of the progression 4, 2, 1 $\cdots$.

11. In the formula $s = \frac{n}{2}[2a + (n-1)d]$, find d in terms of the other letters.

12. In the formula $s = \frac{a - rl}{1 - r}$, find l in terms of the other letters.

13. Show that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$.

14. Prove that the sum of the first n terms of 1, 3, 5, 7, $\cdots$ is n^2.

15. In an arithmetic progression find d in terms of n, l, and s.

Group B

16. Find the sum of the first 8 terms of $\frac{2}{3}$, 1, $\frac{4}{3}$, $\cdots$.

17. Find the sum of the first 12 terms of 2, $\frac{5}{2}$, 3, $\cdots$.

18. Find the sum of the series 1, $-2x$, $4x^2$, $\cdots$ to 7 terms.

19. The sum of 15 terms of an arithmetic progression is 600. Find the series if the common difference is 5.

20. The sum of 8 numbers in arithmetic progression is 44 and the last one is 8. Find the series.

21. Express in the form of an equation the relation connecting a, b, and c, so that they are in geometric progression.

22. Find the arithmetic mean between $\frac{a^2 - b^2}{a}$ and $\frac{a^2 + b^2}{a}$.

23. Find the sum of all the multiples of 6 which lie between 200 and 300.

24. Find 3 numbers in geometric progression whose sum is 26 and whose product is 216. (Let $\frac{a}{r}$, a, and ar be the three given numbers.)

25. Three numbers whose sum is 36 are in arithmetic progression. If 1, 4, and 43 be added to them respectively, the results are in geometric progression. Find the 3 numbers. (Let $a - d$, a, and $a + d$ be the three given numbers.)

26. The sum of three numbers in arithmetic progression is 21, and the sum of their squares is 155. Find the numbers.

27. The arithmetic mean between two numbers is 39, and the geometric mean is 15. Find the numbers.

Group C

28. Find the sum of $\frac{a+1}{a}, \frac{a+2}{a}, \frac{a+3}{a}, \cdots$ to a terms.

29. In the geometric progression $\sqrt{2}, \sqrt[3]{2}, \sqrt[6]{2}, \cdots$, find the 4th term.

30. Find the thirteenth term of the series $\frac{\sqrt{2}-1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}+1}{2}, \cdots$.

31. Prove that the sum of the first n integers plus the sum of the first $n + 1$ integers always gives a result which is a perfect square.

32. Two numbers differ by 50 and their arithmetic mean exceeds their geometric mean by 5. Find the numbers.

33. The product of four terms of a geometric progression is 4 and the fourth term is 4. Find the series.

34. Find the common difference of the arithmetic progression whose first term is 3, and whose second, fourth, and eighth terms are in geometric progression.

35. In a geometric progression the sum of the first and second terms is 12; the sum of the third and fourth terms is 108. Find (*a*) the common ratio and (*b*) the sum of the first 7 terms.

36. Insert two numbers between 6 and 16 so that the first three terms of the series shall be in arithmetic progression and the last three in geometric progression.

37. A and B are 279 miles apart and start at the same time to ride toward each other. A rides 5 miles the first day, 7 miles the second, 9 miles the third, and so on. B rides 30 miles the first day, 27 the second, 24 the third, and so on. After how many days do they meet?

Progress with Progressions

The Babylonian tablets (about 2000 B.C.) and certain problems in the Ahmes Papyrus (about 1700 B.C.) clearly indicate that the ancients were well versed in the progressions. The early knowledge of series, however, was used only as a recreational pastime and no practical value was made of it except in so far as to solve puzzle problems.

While Archimedes (about 225 B.C.) had summed the infinite series $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} \cdots$, the formula for the sum of an infinite geometric series is attributed to Vieta (about 1590).

CUMULATIVE REVIEW

Chapters XIV, XV, and XVI

1. Which of these statements are true? Which are false?

(*a*) If from the top of a cliff 500 feet high the angle of depression of a boat is 32° 15′, the distance of the boat from the base of the cliff is less than 500 feet.

(*b*) In the isosceles triangle in which $a = b$, if angle C is acute, then log tan A is negative.

(*c*) In triangle ABC, if b and c are constant in length, then as angle A increases from 0° to 180° the maximum area triangle ABC can have is when $A = 180°$.

(*d*) The area of a triangle is determined if the measurements of any three of its six parts are given.

(*e*) The area of parallelogram $ABCD$ can be determined if the measurements for angle A, angle B, and side AB are given.

(*f*) If $l = a + (n - 1)d$, then $n = \dfrac{l - a}{d} + 1$.

(*g*) The series $\frac{2}{3}, \frac{1}{3}, \frac{1}{6}, \cdots$ is a decreasing arithmetic progression.

(*h*) The geometric mean between $\frac{2}{3}$ and $\frac{1}{6}$ is $\pm \frac{1}{3}$.

(*i*) The arithmetic mean between $\sin x$ and $2 \sin x$ is $\sin \dfrac{3x}{2}$.

2. Complete each of the following statements:

(*a*) In oblique triangle ABC, if $a = 40$, $b = 30$, and $A = 75°$, then $\sin B = ?$

(*b*) In oblique triangle ABC, if $b = 6$, $c = 4$, and $A = 77°$, then $a^2 = ?$

(*c*) If two sides of a triangle are 15 inches and 25 inches and include an angle of 35°, then the area of the triangle is ? .

(*d*) If a and b are adjacent sides of a parallelogram and C the included angle, then the area of the parallelogram in terms of a, b, and a function of C is ? .

(*e*) If the area of an acute triangle is 132 while two of its sides are 24 and 22, then the number of degrees in the angle included between these two sides is ? .

(*f*) The fifth term of the series $\frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \cdots$ is ? .

(*g*) The sum to infinity of the series $-\frac{1}{2}, \frac{1}{16}, -\frac{1}{128}, \cdots$ is ? .

(*h*) The positive number that must be added to the second number of the set 4, 9, 36, in order that the resulting numbers will form a geometric progression is ? .

(*i*) In an arithmetic progression, if the first term is 2 and the third term is -2, then the fourth term is ? .

3. Using logarithms, find angles B and C of oblique triangle ABC in which $\angle A = 69° 18'$, $b = 5007$ feet, and $c = 4326$ feet.

4. To determine the distance between two objects separated by a swamp, a point is chosen 205.7 feet from one of the objects and 149.7 feet from the other. The angle formed at this point by lines to the objects is $56° 40'$. What is the distance between the objects?

5. Two consecutive sides of a parallelogram measure 12.5 inches and 25.0 inches and include an angle of $41° 17'$. Find the area of the parallelogram.

6. Find the area of a rhombus each of whose sides is 14.2 inches and whose shorter diagonal is 10.9 inches.

7. The sides of a triangle are 72, 135, and 153 respectively. Find the radius of its inscribed circle.

8. By the use of logarithms find the 12th term of the geometric progression in which $a = 278$ and $r = \frac{1}{2}$.

9. The sum of a certain number of consecutive integers is 45. The largest integer is twice the smallest. Find the smallest integer. (Solve as an arithmetic progression.)

10. A certain ball falling from any height rebounds to $\frac{3}{4}$ of that height. What total distance will it have traveled when it strikes the ground for the fifth time if it first falls from a height of 80 feet?

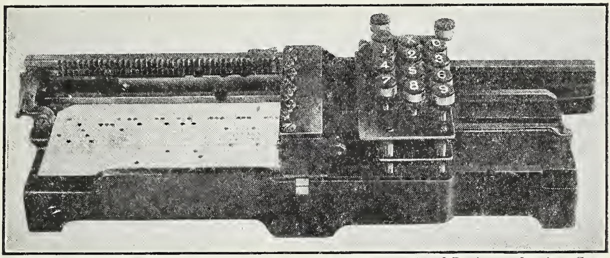
11. Change the repeating decimal $.246363 \cdots$ to an equivalent fraction.

12. James saves 25 cents the first week and 5 cents more each week than he did the preceding week. In how many weeks will he have saved a total sum of \$73.75.

CHAPTER XVII. THE MEANING AND USE OF STATISTICS

All in all, it certainly appears that the rudiments of sound statistical sense are coming to be an essential of a liberal education.

—R. G. WOODWORTH.

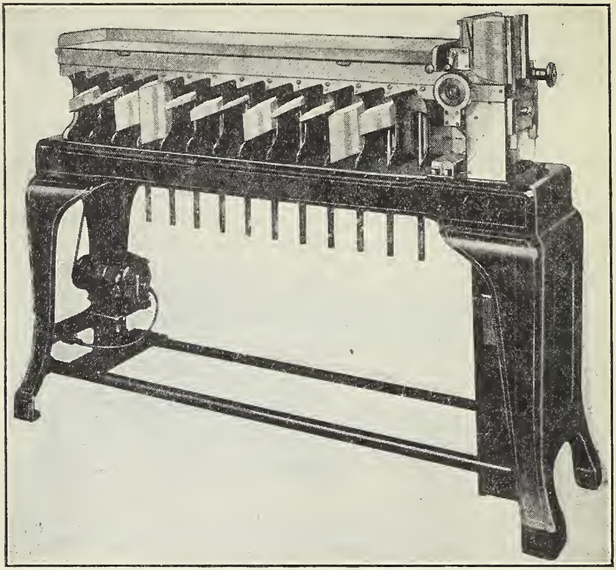


International Business Machines Corp.

Many machines have been invented for use in statistical work. The key punch shown above is used to record data on a card by punching holes. On the card shown, the first hole on the left records the month; the next two the day; the next two the year, and so on.

In order to solve the problems of science and industry, it is very frequently necessary to record a large number of facts and measurements. For example, scientists record the results of hundreds of experiments; governments record births and deaths, the number of people in each industry, the number of people in school, etc.; business men record their sales, their profits and losses; insurance companies keep careful records of the age at which people die so that they can determine insurance rates.

These recorded facts or data are often arranged in tabular form. It is not easy, however, for the human mind to visualize



International Business Machines Corp.

This electric sorting machine can sort from 350 to 400 cards a minute. The punched cards are placed in the holder at the right; the machine is set for the classification desired; and the sorting proceeds automatically. Other machines reproduce the records punched on the cards.

all the meanings and significant relationships in a large array of facts arranged in tabular form. Therefore mathematical and graphic procedures have been scientifically devised whereby collections of facts can be expressed in numbers. The science of dealing with large collections of facts is called *statistics*. The higher mathematics which is the basis of the study of statistical work is beyond the scope of this book, but a brief exposition will whet the appetite of those who wish to continue with this intriguing study.

Studying a set of marks of an examination. After marking a set of papers, a teacher would naturally record the marks in the same order in which he happened to pick up the papers.

For instance, an algebra teacher found that the results of a test yielded the following percentages:

85, 70, 85, 70, 55, 70, 75, 60, 85, 80, 70, 70, 95, 75, 75, 70, 60, 90, 75, 65, 90, 65, 55, 75, 60, 80, 65, 80, 50, 75, 65, 70, 70, 65, 80.

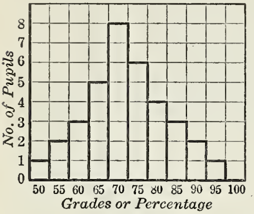
Now here are thirty-five percentages. What do these grades tell us? The answer to this question can be obtained more readily if we arrange the grades in order of excellence, so that we place first the pupil who has the highest percentage, second the pupil who has the next highest percentage, and so on down to the pupil who has the lowest result. Our array of numbers would then be:

95, 90, 90, 85, 85, 85, 80, 80, 80, 80, 75, 75, 75, 75, 75, 75, 70, 70, 70, 70, 70, 70, 70, 70, 65, 65, 65, 65, 65, 60, 60, 60, 55, 55, 50.

This arrangement in *order of rank*, or *rank order*, is called a *rank list.* This rank list enables us to group the number of those attaining the same grade, thereby aiding us in understanding the quality of work done by the class. For instance, we can readily see that the highest grade is 95, and the lowest 50, a difference of 45. This difference is called the *range* or *spread* of the grades. Also from this we can see how many pupils attained 100%, how many 95%, etc., and thus form a *frequency table*, which is a convenient arrangement based on the rank order of attainment. Thus:

Score on Test	50	55	60	65	70	75	80	85	90	95	100
Number of Pupils	1	2	3	5	8	6	4	3	2	1	0

You have already learned how to make a graph of data of this kind. Representing this data in the form of a bar graph, we obtain the accompanying diagram, called a *histogram*, which gives a more vivid picture of the distribution of the percentages of the class than the table.



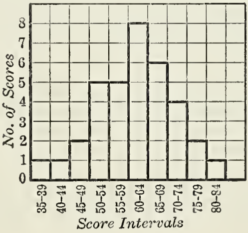
Let us take another example. A teacher found that his students in a geometry class made the following scores:

58, 53, 60, 62, 67, 79, 59, 57, 64, 67, 65, 62, 58, 68, 48, 47, 70, 77, 63, 61, 67, 64, 70, 51, 43, 53, 58, 50, 72, 64, 71, 53, 68, 82, 37.

You will notice that the scores have a very wide range or spread (from 37 to 82), and therefore, to condense the groups, it will be more convenient to tabulate the scores into new arbitrary groups. Now, if we agree to group by 5's, all those who attained a score from 40 to 44 will fall into one group, those from 45 to 49 into another, etc. This agreed interval, 40 to 44, 45 to 49, etc., is frequently termed the *score interval* or *class interval*. It should be remembered that the score interval 40 to 44 includes 40, 41, 42, 43, 44. Thus we have the frequency table.

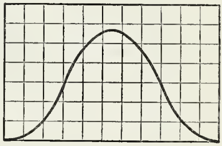
Score Intervals	Number of Scores (Frequency)
35–39	1
40–44	1
45–49	2
50–54	5
55–59	5
60–64	8
65–69	6
70–74	4
75–79	2
80–84	1

Constructing a histogram we have:



The normal frequency curve. The two examples just considered illustrate a common tendency in the measures of mental and physical characteristics of human beings. From the histograms you may observe the tendency for the scores to be more frequent in the central intervals; that is, the scores tend to group about some central score or *point of central tendency.* The frequency of the scores diminishes as the distance from this point of central tendency becomes greater.

In the two examples explained above, the frequency of the scores was not the same on the left side as on the right side of this point of central tendency. Hence the histograms were not symmetrical. However, it may happen, especially when there is a much greater number of items under consideration, that the distribution is such that the frequency in corresponding intervals is the same on the right side as on the left side of this point of central tendency. Then the histogram is symmetrical and the curved graph formed by joining the mid-points of the top line of each bar of the histogram would have a bell-shaped appearance as in the adjoining figure.



This same tendency occurs in most measurements in nature and industry: for example, in the measurement of leaves, in the distribution of wages, in the fluctuations in prices, and the like. Similarly, when we measure mental abilities, as abilities in reading, spelling, computing, or any other form of mental effort, we find that very few people have extremely low ability, very few distinctly superior ability, and that the largest number falls in the middle ranges. There seems to be a fundamental law in operation, the graph of which takes the bell-shaped curve shown above and which is called the *normal frequency curve* and sometimes the *probability curve.* This symmetric distribution is commonly found in physical, chemical, and biological measurements.

Measures of Central Tendency

A teacher had two classes in algebra and wished to find out which class was doing the better work. He gave them the same test in the same time limit and under the same conditions. Then he arranged the results in rank order as follows:

Rank Order Table

Second Period Class		Third Period Class	
Pupil by Letter	Percentage	Pupil by Letter	Percentage
A	96	A	95
B	93	B	94
C	87	C	93
D	86	D	88
E	85	E	88
F	83	F	88
G	81	G	84
H	80	H	83
I	78	I	83
J	77	J	82
K	76	K	78
L	75	L	76
M	74	M	73
N	74	N	72
O	74	O	68
P	68	P	66
Q	67	Q	63
R	62	R	60
S	57	S	57
T	57	T	50
U	52	U	49
V	48		

He then arranged the data in a frequency table.

Frequency Table

Second Period Class		Third Period Class	
Percentage Intervals	Number or Frequency	Percentage Intervals	Number or Frequency
95–99	1	95–99	1
90–94	1	90–94	2
85–89	3	85–89	3
80–84	3	80–84	4
75–79	4	75–79	2
70–74	3	70–74	2
65–69	2	65–69	2
60–64	1	60–64	2
55–59	2	55–59	1
50–54	1	50–54	1
45–49	1	45–49	1
Total	22	Total	21

From these tables the teacher endeavored to compare the two classes. This was difficult because there were so many different values scattered throughout the table. If he could have discovered a single number for each class that would have indicated the tendency of the data to center around a definite value, his problem and comparison would have involved little difficulty.

Thus it is important to find a *single number* that represents the *central tendency* about which all the other scores or measures in a table of data seem to cluster. There are three numbers that accomplish this result very effectively — the *mean* or *average*, the *mode*, and the *median*.

The mean or average. The best known measure of central tendency is the *mean* or *average.* It is precisely the average you learned in your work in arithmetic. Simply stated, it is the sum of the separate measures or scores divided by the number of times these measures or scores were taken. Thus, if a man earned \$3.00, \$4.00, \$3.00, \$5.00, \$5.50, and \$3.50 on six successive days, his average or mean daily wage would be found by finding the sum of all his earnings (\$24) and dividing this sum by the number of days he worked (6), giving the result \$4. We may state this method by the formula

$$\text{Average} = \frac{\text{sum of all the measures or scores}}{N},$$

where N is the number of items or measures in the data.

Returning then to the rank order table of the teacher's second period class, the sum of all the percentages is 1630 and the number of measures is 22, the number of pupils in the class. Therefore, the average or mean $= \frac{1630}{22} = 74.09$. In the teacher's third period class the average or mean $= \frac{1590}{21} = 75.71$.

However, when measures have been grouped into a frequency distribution table, the average or mean is calculated by a slightly different method.

A simple example will illustrate the importance of this new procedure. Suppose a commodity is sold at different prices in 5 different towns, as in town A, at a price cf \$2.00 each, town B, at a price of \$2.50 each, town C, at \$2.25, town D, at \$2.15, and town E, at \$2.20. If no statement is made as to quantities, it would then be correct to find the mean price by adding the individual prices and then dividing by 5. Thus:

$$\text{The mean} = \frac{2 + 2.50 + 2.25 + 2.15 + 2.20}{5} = 2.22.$$

However, if we know that 20,000 were sold in town A, only 50 in town B, 10,000 in town C, only 75 in town D, and 12,250 in town E, undue importance has been attached to the small markets in towns B and D. In this case it would be better to *weight* the importance of each market, in order to give a truer picture of the actual conditions.

To obtain the new mean by "weighting," we multiply each price by the corresponding number sold and then divide the sum of these results by the total number sold. Thus the mean equals

$$\frac{(2 \times 20000)+(2.50 \times 50)+(2.25 \times 10000)+(2.15 \times 75)+(2.20 \times 12250)}{(20000 + 50 + 10000 + 75 + 12250)}$$

$$= \frac{89736.25}{42375} = 2.12,$$

which is appreciably lower than our other estimate (2.22) because we have taken into account the small quantities sold in towns B and D.

Now return to the frequency table of the teacher's algebra classes. The individual percentages, having been grouped into percentage intervals, have lost their identity. Therefore, in order to represent any score falling in an interval, we use the midpoint of that particular interval. Thus to represent the percentage 96 which falls in the interval 95 to 99, we use the midpoint (97) of this interval.

Consequently, to find the weighted mean or average we must multiply the midpoint (M) of each percentage interval by the frequency (F) in that interval, and then divide the sum of these products by the total number (N) of frequencies.

The formula may then be written

$$\textbf{Average} = \frac{\textbf{sum of products } (F \times M)}{N}.$$

The following computation will illustrate, in tabular form, the complete method discussed above for determining the average or mean.

Second Period Class				Third Period Class			
Scores	Mid-point M	Number or Frequency F	Product $F \times M$	Scores	Mid-point M	Number or Frequency F	Product $F \times M$
95–99	97	1	97	95–99	97	1	97
90–94	92	1	92	90–94	92	2	184
85–89	87	3	261	85–89	87	3	261
80–84	82	3	246	80–84	82	4	328
75–79	77	4	308	75–79	77	2	154
70–74	72	3	216	70–74	72	2	144
65–69	67	2	134	65–69	67	2	134
60–64	62	1	62	60–64	62	2	124
55–59	57	2	114	55–59	57	1	57
50–54	52	1	52	50–54	52	1	52
45–49	47	1	47	45–49	47	1	47
Totals		22	1629	Totals		21	1582

Average (or mean) $= \frac{1629}{22} = 74.05.$ Average (or mean) $= \frac{1582}{21} = 75.33.$

The mode. The most frequently occurring number in a table of data is called the *mode.* It is the most common or " most stylish " number in a group of numbers. Thus, in the set of numbers 0, 1, 1, 2, 2, 3, 3, 3, 4, 4, 5, since the most often recurring measure is 3, this measure or score is termed the mode.

Therefore, in the teacher's rank order table the most frequently occurring percentage for the second period class is 74, while for the third period class it is 88. Hence, the modes for the two classes are 74 and 88 respectively.

However, when the measures or scores are grouped into a frequency table, the mode may be taken as the midpoint of that interval which contains the greatest frequency.

Thus the mode of the scores in the frequency table (page 547) of the teacher's second period class is 77, because 77 is the midpoint of the interval 75–79 which corresponds to the largest frequency. Similarly, 82 is the mode for the third period class.

You will observe that the mode is dependent to a large extent upon the size of the class interval selected, and for this reason is often an unstable measure of central tendency.

The median or middle number. The middle measure or score in a rank list is called the *median*. It can be found easily by counting from either end of the list, since it is a number so chosen that there are as many numbers above it as below it.

For example, suppose the eleven scores in a rank list are 56, 60, 63, 68, 73, 75, 80, 84, 87, 91, 96. Since the median is the middle number, we must find which number has as many measures or scores above it as below. If we take the 6th number from either end, there will be 5 other measures or scores above and 5 below. Therefore, the median is the 6th number, which is 75. If the number of scores happen to be even, the median is halfway between the two middle measures. Thus in the rank list, 56, 60, 63, 68, 73, 75, 80, 84, 87, 91, 96, 98, we have 12 measures or scores, the two middle measures of which are 75 and 80, since there will be 5 measures below and 5 above them. The median will therefore be half way between 75 and 80, that is, $77\frac{1}{2}$. Again, in the teacher's rank order table there are 22 scores and the median for the second period class will be half way between the 11th and 12th scores, *i.e.*, $\frac{1}{2}(75 + 76) = 75.5$; while in the third period class it will be the 11th score, *i.e.*, 78.

If, however, we wish to calculate the median when the data is arranged in a frequency distribution, a more detailed computation will be necessary. In this case there are several methods of computing the median, varying in difficulty. However, for purposes of this book the following method is adequate for elementary needs. It is to be understood, nevertheless, that there are more accurate methods which you can find in an advanced treatment of this topic.

We shall repeat the frequency table:

Second Period Class		Third Period Class	
Percentage Intervals	Number or Frequency	Percentage Intervals	Number or Frequency
95–99	1	95–99	1
90–94	1	90–94	2
85–89	3	85–89	3
80–84	3	80–84	4
75–79	4	75–79	2
70–74	3 - - - - - - 10	70–74	2 - - - - - - 9
65–69	2	65–69	2
60–64	1	60–64	2
55–59	2	55–59	1
50–54	1	50–54	1
45–49	1	45–49	1

Since in the second period class there are 22 scores in the distribution, the median therefore is located at the point which lies 11 scores distant from either end of the distribution. If we begin at the lower end, at the score corresponding to the interval 45–49, and add the frequencies in order, 1, 1, 2, 1, etc., 10 scores will take us *through* the percentage interval 70–74. If we proceed to the next percentage interval 75–79, the total number of scores will be $10 + 4 = 14$, 3 too many to give us the required point. To get the 1 score out of these next 4 to make exactly 11 we must, therefore, advance $\frac{1}{4}$ of the next frequency number. In order to obtain the corresponding percentage value of the required point, it will be necessary to take $\frac{1}{4}$ of 5 (the length of the percentage interval) and add this amount ($\frac{1}{4}$ of $5 = 1.25$) to 75, the beginning of the percentage interval in which the required point lies. This takes us exactly 11 scores into the distribution, and gives the value of the median as $75 + 1.25 = 76.25$.

Similarly we can determine the median for the third period class. Here there are 21 scores, and therefore the median is located at that point which lies 10.5 measures or scores distant from either end. Counting from the lower end, exactly 9 scores will take us through the interval 70–74. Therefore, the median must lie in the next interval 75–79. Adjusting the difference as above, the median is found to be 78.75.

The best measure of central tendency. If there are no very extreme measures or scores, the mean is probably the best number to use to represent the central tendency of a group of statistical facts. It is the most familiar measure and is easily calculated.

When the data contains extreme measures or scores, the median, which is not influenced by them, should be used. It too is easily calculated.

However, when the given data contains a single outstanding group of measures, the most satisfactory measure of central tendency is the mode. This is true even though extreme measures may be present. It should be remembered that the mode cannot be used where there is no pronounced group of measures in the data. A few illustrations will make the discussion above clear.

Illustrative examples.

Example 1. A group of students made a collection to purchase some plants in order to decorate the classroom. The teacher subscribed \$1.50; one boy, 45 cents; and 35 other pupils 10¢ each. One pupil, however, was absent when the collection was made and upon his return wished to be a subscriber. He desired to know what his share should be.

Arranging the items in "rank list" form we have, \$1.50, .45, .10, .10, .10, .10, $\cdots$ .10. The average of these items is $\frac{5.45}{37}$ or .15, the mode .10, and the median .10. Evidently the average or mean does not give a very good idea of what should be paid, since there are two extreme measures, \$1.50 and .45. The mode, or "stylish" subscription, conveys a better picture of the amount, because the most often recurring subscription is really sought. Hence the boy was asked to pay 10¢. The value of the median in this particular example happens to be the same as that of the mode. This is not usually the case.

Example 2. A student is to receive his final grade for the term on the basis of the results of seven examinations. His marks were 87%, 83%, 92%, 76%, 83%, 90%, 84%. What should his final grade be?

Arranging in "rank list" order, the marks are 92, 90, 87, 84, 83, 83, 76. The average or mean is $\frac{595}{7} = 85\%$. The mode is 83%, because it occurs twice and any other mark but once. The median or middle mark is 84%, since there are as many items above it as below. Evidently the average or mean indicates the best picture of the ability of the student, since there are no extreme measures which affect the average disproportionately. His final grade is therefore 85%. The number of items is too small to accept the median as reflecting a true picture and the mode is not sufficiently reliable since the mark it represents occurs only twice.

Example 3. The weights of a crew are 212 lb., 190 lb., 196 lb., 200 lb., 205 lb., 202 lb., 206 lb., 195 lb., and the coxswain 110 lb. What is the number which represents the central tendency of the weights?

Evidently the arithmetical average or mean will not truly picture the situation, as the extreme measure, 110 lb., affects the central tendency disproportionately. Again, since there are no two measures alike, there is no mode. The median then is the only measure which will determine, to any degree of truth, the number which represents the central tendency of this group of weights.

Arranging these measures in " rank list " order, we have 212, 206, 205, 202, 200, 196, 195, 190, 110. Since there are nine terms, the median will be the fifth term, 200 pounds.

EXERCISES

1. A student received the following marks on five examinations in algebra: 70%, 75%, 80%, 80%, 40%. Find the average, mode, and median. Which of these numbers do you think gives the best picture of the ability of the boy?

2. In a class of 35 pupils the following frequency table of the heights of the group was determined

Number	2	5	6	9	7	4	2
Height in inches	58–58.9	59–59.9	60–60.9	61–61.9	62–62.9	63–63.9	64–64.9

Draw the frequency histogram. Find the mean, mode, and median.

3. Find the average, mode, and median of the following numbers: 15, 30, 47, 5, 15, 11, 23, 15, 28, 47.

4. A jury was charged to decide the amount of money a defendent should pay for damage to an automobile. The foreman of the jury requested each juror to write on a slip of paper the amount which he considered just. Collecting these slips the foreman tabulated the following amounts: \$200, \$200, \$250, \$175, \$200, \$700, \$125, \$200, \$50, \$225, \$300, \$200. Which of the three measures of central tendency do you think would represent the fairest amount of damages upon which the jury can agrce? Find the average, mode, and median of the amounts suggested by the jurors individually.

5. Some men wished to contribute towards a testimonial gift to the president of their society. Two men contributed \$15 each, five \$5 each, twenty \$3 each, thirty \$1 each, fifty 50¢ each, and sixty-five 25¢ each. Find the average, mode, and median. If another man had wished to contribute, which of the numbers found would have best represented the amount of his contribution. Justify your selection.

6. Two classes in algebra took the same test in fractional equations. The teacher tabulated the results as follows:

Pupils in 1st class	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
Number Right	0	3	4	4	8	5	3	9	7	7	6	10	4	10	3	6	7	4	8	5	1	2
Pupils in 2nd class	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T		
Number Right	1	3	4	2	7	4	8	8	4	6	10	8	3	10	3	9	8	4	9	5		

Using the table as a guide:

(*a*) Make a frequency table for each class, using one example as the class interval, as $0-.99$, $1-1.99$, etc.

(*b*) Make a frequency histogram of each table.

(*c*) Find the average, mode, and median for each class.

(*d*) Which class do you think did better on the test? Justify your conclusions.

Measures of Variability

Suppose that a teacher gave the same algebra test to two classes, one having 34 boys and the other 34 girls. After calculating the average score of each class he found that the boys attained 69.2% and the girls 69%. Therefore, as far as the average scores were concerned, there was no apparent difference in the performance of the two classes. However, upon examining the original scores more closely, he found that the boys' scores ranged from 30% to 92%, and the girls' scores ranged from 40% to 85%. The boys' scores, therefore, seemed to be more variable than the girls' because their range of abilities differed more than that of the girls'. This greater *variability* or *dispersion* may be of much more import than the lack of difference in the average scores would seem to indicate.

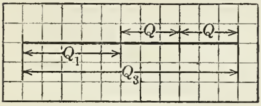
In order to express these characteristics quantitatively, four measures have been devised to take account of this factor of variability or dispersion of a group of scores, namely, the range, the quartile deviation, the average deviation, and the standard deviation.

The range. The *range* is the simplest possible measure of the dispersion of scores because it is the interval or difference between the greatest and least measures. In the above illustration, the range of the boys' scores is $92-30$ or 62, and the range of the girls' scores is $85-40$ or 45. The range is the most general measure of variability or dispersion within a set of measures, and, except for a rough comparison, is unreliable, because it takes into account only the upper and lower limits of the data, and does not recognize the manner in which the scores within these limits are distributed. The range is unduly influenced by extreme measures.

The quartile deviation or Q. If a point is determined in the measures or scores such that one quarter or 25% of all the measures lie below this point, and three quarters or 75% lie above this point, then the value of this point is called the *lower quartile* or *25th percentile* and is usually represented by Q_1. Similarly, if a point is determined such that three-quarters or 75% of the measures lie below this point and one-quarter or 25% lie above this point, then the value of this point is called the *upper quartile* or *75th percentile* and is represented by Q_3. Now since the median is the measure above and below which lie equal numbers of measures or scores, then the two quartiles and the median divide the range into four classes of equal frequency. Therefore, if in a set of data the scores are distributed symmetrically, the values of the median and the quartiles have the following relationship:

$$\text{Median} - Q_1 = Q_3 - \text{Median}.$$

Either difference may be taken as a measure of dispersion. But as no distribution is absolutely symmetrical, it is usual to take as a measure of dispersion (Q), *the quartile deviation*, which is the interval equal to half the interval from Q_1 to Q_3. Thus as the adjoining diagram shows:



$$Q = \frac{Q_3 - Q_1}{2}.$$

Thus, in order to find Q, we must first calculate Q_1 and Q_3, the arithmetical calculations of which are exactly the same as those used in computing the median. The following table will serve as an illustration of the calculation of Q_1, Q_3, and Q.

Fourth Period Class	
Percentage Intervals	Number or Frequency
95–99	1
90–94	2
85–89	2
80–84	2
75–79	6
70–74	8
65–69	6
60–64	5
55–59	3
50–54	2
45–49	1
Total	38

First to find Q_1, we must count off $\frac{1}{4}$ of the total number of measures or scores, *i.e.*, $\frac{1}{4}$ of $38 = 9.5$, from the lower end of the distribution.

Adding the first three frequencies we obtain 6 and the next interval (60–64) contains 5 scores, making a total of 11. But this is too much since we need only 3.5 additional scores above the 6, to make up the necessary 9.5. Therefore, we take $\frac{3.5}{5} \times 5$ (the class interval) or 3.5 and add this amount to 60, the beginning of the next class interval in which Q_1 lies. Thus Q_1 is $60 + 3.5$ or 63.5.

Again count off $\frac{3}{4}$ of the total measures or scores, *i.e.*, $\frac{3}{4}$ of 38 or 28.5, from the lower end of the distribution. Adding the first six frequencies we obtain 25, and the next interval (75–79) contains 6 scores, making a total of 31. But this is too much since we need only 3.5 additional scores above the 25 to make up the necessary 28.5.

We therefore take $\frac{3.5}{6} \times 5$ (the class interval), *i.e.*, $\frac{17.5}{6}$ or 2.92, and add this amount to 75, the beginning of the next class interval in which Q_3 lies. Thus Q_3 is $75 + 2.92$ or 77.92.

Now $$Q = \frac{Q_3 - Q_1}{2}.$$

Substituting

$$Q = \frac{77.92 - 63.5}{2} = 7.21.$$

Q, the quartile deviation, is of considerable importance, because it actually measures the average distance of the two quartile points, Q_1 and Q_3, from the median.

The average deviation. Suppose we have the following measures of scores: 12, 16, 20, 24, and 28. We have learned that the average or mean of these numbers is the sum of all these scores divided by 5, *i.e.*, $\frac{100}{5}$ or 20. The *deviation* of each individual score is the amount by which that score differs from the average or mean. Thus finding these deviations, we have 8, 4, 0, 4, 8, the sum of which is 24. If we divide this sum by 5, the number of scores, we have $24 \div 5$ or 4.8. This number 4.8 is called the *average deviation* and is usually represented by AD.

In general, however, the average deviation or AD is the average of the deviations of all the individual measures in a set computed from any one of the three numbers of central tendency. Usually the average deviation is computed from the mean or average. Observe that the signs of the deviations are not considered in computing the AD.

The formula is, therefore,

$$AD = \frac{\text{Sum of deviations of all scores}}{N},$$

where N represents the number of items or measures in the data.

However, when the scores are grouped into class intervals, it is impossible to get the deviation of each score individually from the average or mean. In this case we calculate the deviations from the midpoint of each class interval, and multiply by the respective frequencies.

$$\therefore AD = \frac{\text{Sum of products } (F \times D)}{N},$$

where F represents the frequency, D the deviation, and N the total number of frequencies. In general, a large value for AD indicates that the scores in the distribution are dispersed or scattered about the number of central tendency, while a small value for AD indicates that they are bunched or concentrated within comparatively small limits.

The standard deviation. Let us consider the following measures or scores: 12, 16, 20, 24, and 28, the average or mean of which is 20. In finding the AD, you will remember that all the deviations were taken without regard to sign. If, however, we take the signs into consideration, and find the deviations from the average or mean by subtracting it from each measure, we have

$$-8, -4, 0, +4, +8.$$

Squaring each deviation above, we have

$$+64, +16, 0, +16, +64.$$

Computing the average of these squares, we obtain

$$\text{Average} = \frac{+64 + 16 + 0 + 16 + 64}{5} = \frac{160}{5} = 32.$$

Extracting the square root of this average, we get finally

$$\sqrt{32} = 5.656 \text{ or } 5.66.$$

This last number is called the *standard deviation*, and is usually represented by SD or σ.

The formula for this new measure of variability is then:

$$SD \text{ (or } \sigma) = \sqrt{\frac{\text{Sum of the squares of deviations}}{N}}$$

where N represents the number of items or measures in the data.

It can be shown in a more detailed treatise in statistics that the standard deviation is the most reliable of the measures of dispersion, because it is, in general, less influenced by chance fluctuations than the average deviation.

However, when the scores are grouped into class intervals, the process is identical with that outlined above for ungrouped measures, except that in addition to squaring the deviations of the midpoint of each class interval from the average or mean, we " weight " each result by multiplying each of these squared deviations by its respective frequency.

The calculation of the AD and SD, as outlined above, is illustrated by the following example.

Illustrative example.

Find the AD and SD of the data for the second period class.

Scores	M Midpoint	F Frequency	D Deviation	$F \times D$	$F \times D^2$
95–99	97	1	22.95	22.95	526.70
90–94	92	1	17.95	17.95	322.20
85–89	87	3	12.95	38.85	503.11
80–84	82	3	7.95	23.85	189.61
75–79	77	4	2.95	11.80	34.81
70–74	72	3	− 2.05	− 6.15	12.61
65–69	67	2	− 7.05	− 14.10	99.41
60–64	62	1	− 12.05	− 12.05	145.20
55–59	57	2	− 17.05	− 34.10	581.41
50–54	52	1	− 22.05	− 22.05	486.20
45–49	47	1	− 27.05	− 27.05	731.70
	Totals	22		230.90	3632.96

Average = 74.05.

$$AD = \frac{\text{Sum of products } (F \times D)}{N} = \frac{230.90}{22} = 10.49.$$

$$SD = \sqrt{\frac{\text{Sum of products } (F \times D^2)}{N}} = \sqrt{\frac{3632.96}{22}}$$
$$= \sqrt{165.1345} = 12.85.$$

EXERCISES

1. The following distribution represents the scores made on a test for entrance to college:

Number	2	4	10	8	17	27	17	12	10	8	7	6
Scores	24–28	29–33	34–38	39–43	44–48	49–53	54–58	59–63	64–68	69–73	74–78	79–83

Find the average, mode, median, Q, AD, and SD.

2. Find the average, median, Q, AD, and SD of each of the following frequency distributions:

(a) *Scores*	*F*	(b) *Scores*	*F*	(c) *Scores*	*F*
95–99	1	95–99	1	95–99	2
90–94	1	90–94	1	90–94	2
85–89	2	85–89	2	85–89	2
80–84	3	80–84	4	80–84	4
75–79	7	75–79	3	75–79	5
70–74	9	70–74	6	70–74	9
65–69	5	65–69	4	65–69	6
60–64	4	60–64	4	60–64	3
55–59	0	55–59	2	55–59	4
50–54	2	50–54	0	50–54	2
45–49	4	45–49	2	45–49	0
40–44	1	40–44	1	40–44	1

3. The table below gives the height frequency distribution of the members of a freshman class in a certain college:

Number	Height
1	$57\text{–}57\frac{7}{8}$
2	$58\text{–}58\frac{7}{8}$
3	$59\text{–}59\frac{7}{8}$
4	$60\text{–}60\frac{7}{8}$
7	$61\text{–}61\frac{7}{8}$
12	$62\text{–}62\frac{7}{8}$
32	$63\text{–}63\frac{7}{8}$
52	$64\text{–}64\frac{7}{8}$
74	$65\text{–}65\frac{7}{8}$
88	$66\text{–}66\frac{7}{8}$
89	$67\text{–}67\frac{7}{8}$
91	$68\text{–}68\frac{7}{8}$
89	$69\text{–}69\frac{7}{8}$
75	$70\text{–}70\frac{7}{8}$
47	$71\text{–}71\frac{7}{8}$
25	$72\text{–}72\frac{7}{8}$
11	$73\text{–}73\frac{7}{8}$

Find the average, mode, median, Q, and SD of the distribution.

Correlation. Frequently it is of great importance to discover the relation, if any, between different abilities. For example, the question may arise as to whether there is any relation between general intelligence as measured by an intelligence test and work in school as measured by "marks." More specifically it is often necessary to know whether the ability in a test in one subject has any relation to the ability in a test in another subject.

That part of the study of statistics which treats of the methods of discovering these relationships is called *correlation.*

The following illustrations will serve to explain further the underlying meaning of the term "correlation."

1. We know that the diameter of a circle is always twice its radius, no matter how large or small the circle may be. An increase or decrease in the radius of a circle will give a corresponding increase or decrease in the diameter by just twice that amount. This relationship is fixed and definite, and we say that the correlation between the radius and diameter of a circle is perfect. This perfection in relationship we represent by 1.

2. A class of 35 students takes tests in algebra and physics, and the results in rank order show that the same arrangement of abilities is maintained throughout the list, that is, the pupil first in algebra is first in physics; the second in algebra is rated the second in physics; etc. It is evident that each student has relatively the same position in one test as in the other. In other words, the correlation between the two lists is perfect, and therefore equal to 1.

3. If the rank order lists in Example 2 showed that the pupil first in the algebra test was 7th in the physics test, the second in the algebra test ranked 31st in the physics test, the 15th in the algebra test was first in the physics test, etc., it is evident that there is no correspondence between the amounts of capacity or abilities exhibited by the same individual in the two tests. There is just no correlation present, which fact is represented by 0.

4. If, however, the rank order lists in Example 2 showed that the pupil first in the algebra was last or 35th in the physics test; the second in the algebra test was 34th in the physics test; the third in the algebra test was 33rd in the physics; $\cdots$ and the last in the algebra test was first in the physics test; it is evident that a high degree of ability in the algebra test may be associated with a low degree of ability in the physics test, and vice versa. The relation here is fixed and definite enough but is opposite or *inverse.* This perfect inverse relationship or correlation is represented by -1.

These numbers representing the degree of correlation are called *coefficients of correlation* and range from -1 through 0 to $+1$. Of course, between these limits we may have relations of varying degree indicated by such coefficients as $-.9$, $-.8$, $\cdots$ 0, .1, .2, .3, $\cdots$ .9. It should be remembered, however, that a *positive* relation or correspondence is indicated by a *positive* correlation coefficient; *absence* of relation, by a *zero* correlation coefficient; an *inverse* relation by a *negative* correlation.

The different methods by which these coefficients of correlation may be computed and their uses are beyond the scope of this book, but those students who are sufficiently interested are referred to any standard text on statistics.

Who's Who in Statistics?

Adolphe Quetelet (1796–1874) has been termed the "founder of modern statistics" because he was the first to establish official statistical societies and statistical offices, and because he made extensive researches on the application of probability to the physical and social sciences.

To Karl Pearson are due the terms *mode* and *standard deviation.* He also made extensive research into the applications of statistical theory to the sciences.

CUMULATIVE REVIEW

Chapters XV, XVI, and XVII

1. Which of these statements are true? Which are false?

(*a*) In the formula for the area of a triangle, $K = \frac{1}{2} bc \sin A$, K can never be negative.

(*b*) If the legs of an isosceles triangle are constant in length, then the area of the triangle increases as the vertex angle increases.

(*c*) The 7th term of the series $\frac{2}{3}, \frac{1}{3}, \frac{1}{6}, \cdots$ is $\frac{1}{96}$.

(*d*) The only geometric mean between $\tan x$ and $\cot x$ is 1.

(*e*) The last or nth term of the progression 2, 6, 10, $\cdots$ in terms of n is $4n - 2$.

(*f*) The range of a set of measures is the difference between the highest and lowest measures.

(*g*) The median is the most frequently occurring number in a table of data.

(*h*) If there are no very extreme measures or scores, the average is the best number to use to represent the central tendency of a group of statistical facts.

(*i*) The average deviation is the most reliable of the measures of dispersion.

(*j*) The correlation between the diameter and circumference of a circle is 1.

2. Complete each of the following statements:

(*a*) From the formula $K = \frac{1}{2}bc \sin A$ for the area of a triangle, it is evident that the area K is the greatest possible for any given values of b and c, when A contains ? degrees.

(*b*) If x and b are adjacent sides of a parallelogram whose area is P, and if C is their included angle, then the formula for x in terms of b, P, and a function of C is ? .

(*c*) The 91st terms of the series 5, 8, 11, $\cdots$ is ? .

(*d*) .2222 $\cdots$ expressed as a common fraction is ? .

(*e*) The sum of all the odd numbers between 4 and 100 is ? .

(*f*) The average of the deviations of all separate measures in a series taken from a measure of central tendency is called ? .

(*g*) When the scores of a distribution are closely packed together, the quartiles will be close together and Q will be ? (large or small).

(*h*) If a man earns \$3, \$4, \$3$\frac{1}{2}$, \$5, and \$4$\frac{1}{2}$ on five successive days, his average daily wage is ? .

(*i*) If the marks of a class of 21 pupils are arranged in rank list order, the 11th or middle mark is called the ? .

(*j*) Coefficients of correlation may range from ? to ? .

3. The diagonal of a rectangle is 24.6 inches long and makes an angle of 63° 22′ with the shorter side of the rectangle. Find the area of the rectangle.

4. At a point P on a straight road extending east and west, a surveyor runs a line 700 feet long in a direction north of east to a point A, making an angle of 42° with the road. From A he continues the line due east for 625 feet to a point B, then due south to a point C on the road, finally returning from C to P. Find the area of the tract of land $PABC$.

5. A ball falls from a height of 48 feet and at every fall rebounds $\frac{2}{3}$ of the distance. What is the total distance passed through by the ball before it comes to rest?

6. The following tabulation represents the number of examples right in a certain timed test for each of two classes. Find the median and mode for each class.

Class 1		Class 2	
Names	Number Right	Names	Number Right
A	24	P	23
B	21	Q	20
C	20	R	19
D	18	S	17
E	17	T	16
F	16	U	15
G	15	V	14
H	14	W	13
I	14	X	13
J	14	Y	13
K	13	Z	12
L	13	AA	12
M	12	BB	11
N	12	CC	11
O	9	DD	10

7. Find three numbers in geometric progression whose sum is 13 and whose product is 27.

8. Construct a frequency polygon if a class of 35 made the following percentages in an examination.

Number of Pupils	Per Cent
5	90–99
9	80–89
13	70–79
6	60–69
0	50–59
2	40–49

9. The frequency distribution of percentages of a geometry class on a certain test is given in the following table:

Number	Percentages
2	95–99
2	90–94
2	85–89
4	80–84
5	75–79
9	70–74
6	65–69
3	60–64
4	55–59
2	50–54
1	45–49

Find the average, mean, Q, AD, and SD.

10. The frequency distribution of weights, to the nearest pound, of adult males in a certain community is given by the following table:

Weight in Pounds	Number of Men
89 to 98	2
99 to 108	26
109 to 118	131
119 to 128	338
129 to 138	694
139 to 148	1240
149 to 158	1075
159 to 168	884
169 to 178	492
179 to 188	304
189 to 198	174
199 to 208	75
209 to 218	62
219 to 228	34
229 to 238	10
239 to 248	9

Find the average, mode, mean, Q, AD, and SD.

11. Find three numbers in arithmetic progression such that the sum of the first and third is 24 and the product of the first and second is 96.

12. The wealth of a community is represented by the following table:

Number of Persons	Wealth per Person
3	$25,000
6	10,000
15	7,000
30	4,000
45	3,500
93	400
62	100
31	0

Find the average deviation from the median.

LOGARITHMIC AND TRIGONOMETRIC TABLES

TABLE I. — Common Logarithms of Numbers *

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

* This table gives the mantissas of numbers with the decimal point omitted in each case. Characteristics are determined by inspection from the numbers.

TABLE I. — COMMON LOGARITHMS OF NUMBERS (*Concl.*)

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

TABLE II. — Values of Trigonometric Functions

Angle	Sin	Cos	Tan	Cot	
0° 00′	.0000	1.0000	.0000	—	**90° 00′**
10	.0029	1.0000	.0029	343.77	50
20	.0058	1.0000	.0058	171.89	40
30	.0087	1.0000	.0087	114.59	30
40	.0116	.9999	.0116	85.940	20
50	.0145	.9999	.0145	68.750	10
1° 00′	.0175	.9998	.0175	57.290	**89° 00′**
10	.0204	.9998	.0204	49.104	50
20	.0233	.9997	.0233	42.964	40
30	.0262	.9997	.0262	38.188	30
40	.0291	.9996	.0291	34.368	20
50	.0320	.9995	.0320	31.242	10
2° 00′	.0349	.9994	.0349	28.636	**88° 00′**
10	.0378	.9993	.0378	26.432	50
20	.0407	.9992	.0407	24.542	40
30	.0436	.9990	.0437	22.904	30
40	.0465	.9989	.0466	21.470	20
50	.0494	.9988	.0495	20.206	10
3° 00′	.0523	.9986	.0524	19.081	**87° 00′**
10	.0552	.9985	.0553	18.075	50
20	.0581	.9983	.0582	17.169	40
30	.0610	.9981	.0612	16.350	30
40	.0640	.9980	.0641	15.605	20
50	.0669	.9978	.0670	14.924	10
4° 00′	.0698	.9976	.0699	14.301	**86° 00′**
10	.0727	.9974	.0729	13.727	50
20	.0756	.9971	.0758	13.197	40
30	.0785	.9969	.0787	12.706	30
40	.0814	.9967	.0816	12.251	20
50	.0843	.9964	.0846	11.826	10
5° 00′	.0872	.9962	.0875	11.430	**85° 00′**
10	.0901	.9959	.0904	11.059	50
20	.0929	.9957	.0934	10.712	40
30	.0958	.9954	.0963	10.385	30
40	.0987	.9951	.0992	10.078	20
50	.1016	.9948	.1022	9.7882	10
6° 00′	.1045	.9945	.1051	9.5144	**84° 00′**
10	.1074	.9942	.1080	9.2553	50
20	.1103	.9939	.1110	9.0098	40
30	.1132	.9936	.1139	8.7769	30
40	.1161	.9932	.1169	8.5555	20
50	.1190	.9929	.1198	8.3450	10
7° 00′	.1219	.9925	.1228	8.1443	**83° 00′**
10	.1248	.9922	.1257	7.9530	50
20	.1276	.9918	.1287	7.7704	40
30	.1305	.9914	.1317	7.5958	30
40	.1334	.9911	.1346	7.4287	20
50	.1363	.9907	.1376	7.2687	10
8° 00′	.1392	.9903	.1405	7.1154	**82° 00′**
10	.1421	.9899	.1435	6.9682	50
20	.1449	.9894	.1465	6.8269	40
30	.1478	.9890	.1495	6.6912	30
40	.1507	.9886	.1524	6.5606	20
50	.1536	.9881	.1554	6.4348	10
9° 00′	.1564	.9877	.1584	6.3138	**81° 00′**
	Cos	**Sin**	**Cot**	**Tan**	**Angle**

TABLE II. — VALUES OF TRIGONOMETRIC FUNCTIONS (*Cont.*)

Angle	Sin	Cos	Tan	Cot	
9° 00′	.1564	.9877	.1584	6.3138	**81° 00′**
10	.1593	.9872	.1614	6.1970	50
20	.1622	.9868	.1644	6.0844	40
30	.1650	.9863	.1673	5.9758	30
40	.1679	.9858	.1703	5.8708	20
50	.1708	.9853	.1733	5.7694	10
10° 00′	.1736	.9848	.1763	5.6713	**80° 00′**
10	.1765	.9843	.1793	5.5764	50
20	.1794	.9838	.1823	5.4845	40
30	.1822	.9833	.1853	5.3955	30
40	.1851	.9827	.1883	5.3093	20
50	.1880	.9822	.1914	5.2257	10
11° 00′	.1908	.9816	.1944	5.1446	**79° 00′**
10	.1937	.9811	.1974	5.0658	50
20	.1965	.9805	.2004	4.9894	40
30	.1994	.9799	.2035	4.9152	30
40	.2022	.9793	.2065	4.8430	20
50	.2051	.9787	.2095	4.7729	10
12° 00′	.2079	.9781	.2126	4.7046	**78° 00′**
10	.2108	.9775	.2156	4.6382	50
20	.2136	.9769	.2186	4.5736	40
30	.2164	.9763	.2217	4.5107	30
40	.2193	.9757	.2247	4.4494	20
50	.2221	.9750	.2278	4.3897	10
13° 00′	.2250	.9744	.2309	4.3315	**77° 00′**
10	.2278	.9737	.2339	4.2747	50
20	.2306	.9730	.2370	4.2193	40
30	.2334	.9724	.2401	4.1653	30
40	.2363	.9717	.2432	4.1126	20
50	.2391	.9710	.2462	4.0611	10
14° 00′	.2419	.9703	.2493	4.0108	**76° 00′**
10	.2447	.9696	.2524	3.9617	50
20	.2476	.9689	.2555	3.9136	40
30	.2504	.9681	.2586	3.8667	30
40	.2532	.9674	.2617	3.8208	20
50	.2560	.9667	.2648	3.7760	10
15° 00′	.2588	.9659	.2679	3.7321	**75° 00′**
10	.2616	.9652	.2711	3.6891	50
20	.2644	.9644	.2742	3.6470	40
30	.2672	.9636	.2773	3.6059	30
40	.2700	.9628	.2805	3.5656	20
50	.2728	.9621	.2836	3.5261	10
16° 00′	.2756	.9613	.2867	3.4874	**74° 00′**
10	.2784	.9605	.2899	3.4495	50
20	.2812	.9596	.2931	3.4124	40
30	.2840	.9588	.2962	3.3759	30
40	.2868	.9580	.2994	3.3402	20
50	.2896	.9572	.3026	3.3052	10
17° 00′	.2924	.9563	.3057	3.2709	**73° 00′**
10	.2952	.9555	.3089	3.2371	50
20	.2979	.9546	.3121	3.2041	40
30	.3007	.9537	.3153	3.1716	30
40	.3035	.9528	.3185	3.1397	20
50	.3062	.9520	.3217	3.1084	10
18° 00′	.3090	.9511	.3249	3.0777	**72° 00′**
	Cos	**Sin**	**Cot**	**Tan**	**Angle**

TABLE II. — Values of Trigonometric Functions (*Cont.*)

Angle		Sin	Cos	Tan	Cot		
18°	**00′**	.3090	.9511	.3249	3.0777	**72°**	**00′**
	10	.3118	.9502	.3281	3.0475		50
	20	.3145	.9492	.3314	3.0178		40
	30	.3173	.9483	.3346	2.9887		30
	40	.3201	.9474	.3378	2.9600		20
	50	.3228	.9465	.3411	2.9319		10
19°	**00′**	.3256	.9455	.3443	2.9042	**71°**	**00′**
	10	.3283	.9446	.3476	2.8770		50
	20	.3311	.9436	.3508	2.8502		40
	30	.3338	.9426	.3541	2.8239		30
	40	.3365	.9417	.3574	2.7980		20
	50	.3393	.9407	.3607	2.7725		10
20°	**00′**	.3420	.9397	.3640	2.7475	**70°**	**00′**
	10	.3448	.9387	.3673	2.7228		50
	20	.3475	.9377	.3706	2.6985		40
	30	.3502	.9367	.3739	2.6746		30
	40	.3529	.9356	.3772	2.6511		20
	50	.3557	.9346	.3805	2.6279		10
21°	**00′**	.3584	.9336	.3839	2.6051	**69°**	**00′**
	10	.3611	.9325	.3872	2.5826		50
	20	.3638	.9315	.3906	2.5605		40
	30	.3665	.9304	.3939	2.5386		30
	40	.3692	.9293	.3973	2.5172		20
	50	.3719	.9283	.4006	2.4960		10
22°	**00′**	.3746	.9272	.4040	2.4751	**68°**	**00′**
	10	.3773	.9261	.4074	2.4545		50
	20	.3800	.9250	.4108	2.4342		40
	30	.3827	.9239	.4142	2.4142		30
	40	.3854	.9228	.4176	2.3945		20
	50	.3881	.9216	.4210	2.3750		10
23°	**00′**	.3907	.9205	.4245	2.3559	**67°**	**00′**
	10	.3934	.9194	.4279	2.3369		50
	20	.3961	.9182	.4314	2.3183		40
	30	.3987	.9171	.4348	2.2998		30
	40	.4014	.9159	.4383	2.2817		20
	50	.4041	.9147	.4417	2.2637		10
24°	**00′**	.4067	.9135	.4452	2.2460	**66°**	**00′**
	10	.4094	.9124	.4487	2.2286		50
	20	.4120	.9112	.4522	2.2113		40
	30	.4147	.9100	.4557	2.1943		30
	40	.4173	.9088	.4592	2.1775		20
	50	.4200	.9075	.4628	2.1609		10
25°	**00′**	.4226	.9063	.4663	2.1445	**65°**	**00′**
	10	.4253	.9051	.4699	2.1283		50
	20	.4279	.9038	.4734	2.1123		40
	30	.4305	.9026	.4770	2.0965		30
	40	.4331	.9013	.4806	2.0809		20
	50	.4358	.9001	.4841	2.0655		10
26°	**00′**	.4384	.8988	.4877	2.0503	**64°**	**00′**
	10	.4410	.8975	.4913	2.0353		50
	20	.4436	.8962	.4950	2.0204		40
	30	.4462	.8949	.4986	2.0057		30
	40	.4488	.8936	.5022	1.9912		20
	50	.4514	.8923	.5059	1.9768		10
27°	**00′**	.4540	.8910	.5095	1.9626	**63°**	**00′**
		Cos	Sin	Cot	Tan	Angle	

TABLE II. — Values of Trigonometric Functions (*Cont.*)

Angle	Sin	Cos	Tan	Cot	
27° 00′	.4540	.8910	.5095	1.9626	**63° 00′**
10	.4566	.8897	.5132	1.9486	50
20	.4592	.8884	.5169	1.9347	40
30	.4617	.8870	.5206	1.9210	30
40	.4643	.8857	.5243	1.9074	20
50	.4669	.8843	.5280	1.8940	10
28° 00′	.4695	.8829	.5317	1.8807	**62° 00′**
10	.4720	.8816	.5354	1.8676	50
20	.4746	.8802	.5392	1.8546	40
30	.4772	.8788	.5430	1.8418	30
40	.4797	.8774	.5467	1.8291	20
50	.4823	.8760	.5505	1.8165	10
29° 00′	.4848	.8746	.5543	1.8040	**61° 00′**
10	.4874	.8732	.5581	1.7917	50
20	.4899	.8718	.5619	1.7796	40
30	.4924	.8704	.5658	1.7675	30
40	.4950	.8689	.5696	1.7556	20
50	.4975	.8675	.5735	1.7437	10
30° 00′	.5000	.8660	.5774	1.7321	**60° 00′**
10	.5025	.8646	.5812	1.7205	50
20	.5050	.8631	.5851	1.7090	40
30	.5075	.8616	.5890	1.6977	30
40	.5100	.8601	.5930	1.6864	20
50	.5125	.8587	.5969	1.6753	10
31° 00′	.5150	.8572	.6009	1.6643	**59° 00′**
10	.5175	.8557	.6048	1.6534	50
20	.5200	.8542	.6088	1.6426	40
30	.5225	.8526	.6128	1.6319	30
40	.5250	.8511	.6168	1.6212	20
50	.5275	.8496	.6208	1.6107	10
32° 00′	.5299	.8480	.6249	1.6003	**58° 00′**
10	.5324	.8465	.6289	1.5900	50
20	.5348	.8450	.6330	1.5798	40
30	.5373	.8434	.6371	1.5697	30
40	.5398	.8418	.6412	1.5597	20
50	.5422	.8403	.6453	1.5497	10
33° 00′	.5446	.8387	.6494	1.5399	**57° 00′**
10	.5471	.8371	.6536	1.5301	50
20	.5495	.8355	.6577	1.5204	40
30	.5519	.8339	.6619	1.5108	30
40	.5544	.8323	.6661	1.5013	20
50	.5568	.8307	.6703	1.4919	10
34° 00′	.5592	.8290	.6745	1.4826	**56° 00′**
10	.5616	.8274	.6787	1.4733	50
20	.5640	.8258	.6830	1.4641	40
30	.5664	.8241	.6873	1.4550	30
40	.5688	.8225	.6916	1.4460	20
50	.5712	.8208	.6959	1.4370	10
35° 00′	.5736	.8192	.7002	1.4281	**55° 00′**
10	.5760	.8175	.7046	1.4193	50
20	.5783	.8158	.7089	1.4106	40
30	.5807	.8141	.7133	1.4019	30
40	.5831	.8124	.7177	1.3934	20
50	.5854	.8107	.7221	1.3848	10
36° 00′	.5878	.8090	.7265	1.3764	**54° 00′**
	Cos	**Sin**	**Cot**	**Tan**	**Angle**

TABLE II. — VALUES OF TRIGONOMETRIC FUNCTIONS (*Concl.*)

Angle	Sin	Cos	Tan	Cot	
36° 00′	.5878	.8090	.7265	1.3764	**54° 00′**
10	.5901	.8073	.7310	1.3680	50
20	.5925	.8056	.7355	1.3597	40
30	.5948	.8039	.7400	1.3514	30
40	.5972	.8021	.7445	1.3432	20
50	.5995	.8004	.7490	1.3351	10
37° 00′	.6018	.7986	.7536	1.3270	**53° 00′**
10	.6041	.7969	.7581	1.3190	50
20	.6065	.7951	.7627	1.3111	40
30	.6088	.7934	.7673	1.3032	30
40	.6111	.7916	.7720	1.2954	20
50	.6134	.7898	.7766	1.2876	10
38° 00′	.6157	.7880	.7813	1.2799	**52° 00′**
10	.6180	.7862	.7860	1.2723	50
20	.6202	.7844	.7907	1.2647	40
30	.6225	.7826	.7954	1.2572	30
40	.6248	.7808	.8002	1.2497	20
50	.6271	.7790	.8050	1.2423	10
39° 00′	.6293	.7771	.8098	1.2349	**51° 00′**
10	.6316	.7753	.8146	1.2276	50
20	.6338	.7735	.8195	1.2203	40
30	.6361	.7716	.8243	1.2131	30
40	.6383	.7698	.8292	1.2059	20
50	.6406	.7679	.8342	1.1988	10
40° 00′	.6428	.7660	.8391	1.1918	**50° 00′**
10	.6450	.7642	.8441	1.1847	50
20	.6472	.7623	.8491	1.1778	40
30	.6494	.7604	.8541	1.1708	30
40	.6517	.7585	.8591	1.1640	20
50	.6539	.7566	.8642	1.1571	10
41° 00′	.6561	.7547	.8693	1.1504	**49° 00′**
10	.6583	.7528	.8744	1.1436	50
20	.6604	.7509	.8796	1.1369	40
30	.6626	.7490	.8847	1.1303	30
40	.6648	.7470	.8899	1.1237	20
50	.6670	.7451	.8952	1.1171	10
42° 00′	.6691	.7431	.9004	1.1106	**48° 00′**
10	.6713	.7412	.9057	1.1041	50
20	.6734	.7392	.9110	1.0977	40
30	.6756	.7373	.9163	1.0913	30
40	.6777	.7353	.9217	1.0850	20
50	.6799	.7333	.9271	1.0786	10
43° 00′	.6820	.7314	.9325	1.0724	**47° 00′**
10	.6841	.7294	.9380	1.0661	50
20	.6862	.7274	.9435	1.0599	40
30	.6884	.7254	.9490	1.0538	30
40	.6905	.7234	.9545	1.0477	20
50	.6926	.7214	.9601	1.0416	10
44° 00′	.6947	.7193	.9657	1.0355	**46° 00′**
10	.6967	.7173	.9713	1.0295	50
20	.6988	.7153	.9770	1.0235	40
30	.7009	.7133	.9827	1.0176	30
40	.7030	.7112	.9884	1.0117	20
50	.7050	.7092	.9942	1.0058	10
45° 00′	.7071	.7071	1.0000	1.0000	**45° 00′**
	Cos	Sin	Cot	Tan	Angle

TABLE III. — Logarithms of Trigonometric Functions*

Angle	L Sin	L Cos	L Tan	L Cot	
0° 00′	—	10.0000	—	—	90° 00′
10	7.4637	10.0000	7.4637	12.5363	50
20	7.7648	10.0000	7.7648	12.2352	40
30	7.9408	10.0000	7.9409	12.0591	30
40	8.0658	10.0000	8.0658	11.9342	20
50	8.1627	10.0000	8.1627	11.8373	10
1° 00′	8.2419	9.9999	8.2419	11.7581	89° 00′
10	8.3088	9.9999	8.3089	11.6911	50
20	8.3668	9.9999	8.3669	11.6331	40
30	8.4179	9.9999	8.4181	11.5819	30
40	8.4637	9.9998	8.4638	11.5362	20
50	8.5050	9.9998	8.5053	11.4947	10
2° 00′	8.5428	9.9997	8.5431	11.4569	88° 00′
10	8.5776	9.9997	8.5779	11.4221	50
20	8.6097	9.9996	8.6101	11.3899	40
30	8.6397	9.9996	8.6401	11.3599	30
40	8.6677	9.9995	8.6682	11.3318	20
50	8.6940	9.9995	8.6945	11.3055	10
3° 00′	8.7188	9.9994	8.7194	11.2806	87° 00′
10	8.7423	9.9993	8.7429	11.2571	50
20	8.7645	9.9993	8.7652	11.2348	40
30	8.7857	9.9992	8.7865	11.2135	30
40	8.8059	9.9991	8.8067	11.1933	20
50	8.8251	9.9990	8.8261	11.1739	10
4° 00′	8.8436	9.9989	8.8446	11.1554	86° 00′
10	8.8613	9.9989	8.8624	11.1376	50
20	8.8783	9.9988	8.8795	11.1205	40
30	8.8946	9.9987	8.8960	11.1040	30
40	8.9104	9.9986	8.9118	11.0882	20
50	8.9256	9.9985	8.9272	11.0728	10
5° 00′	8.9403	9.9983	8.9420	11.0580	85° 00′
10	8.9545	9.9982	8.9563	11.0437	50
20	8.9682	9.9981	8.9701	11.0299	40
30	8.9816	9.9980	8.9836	11.0164	30
40	8.9945	9.9979	8.9966	11.0034	20
50	9.0070	9.9977	9.0093	10.9907	10
6° 00′	9.0192	9.9976	9.0216	10.9784	84° 00′
10	9.0311	9.9975	9.0336	10.9664	50
20	9.0426	9.9973	9.0453	10.9547	40
30	9.0539	9.9972	9.0567	10.9433	30
40	9.0648	9.9971	9.0678	10.9322	20
50	9.0755	9.9969	9.0786	10.9214	10
7° 00′	9.0859	9.9968	9.0891	10.9109	83° 00′
10	9.0961	9.9966	9.0995	10.9005	50
20	9.1060	9.9964	9.1096	10.8904	40
30	9.1157	9.9963	9.1194	10.8806	30
40	9.1252	9.9961	9.1291	10.8709	20
50	9.1345	9.9959	9.1385	10.8615	10
8° 00′	9.1436	9.9958	9.1478	10.8522	82° 00′
10	9.1525	9.9956	9.1569	10.8431	50
20	9.1612	9.9954	9.1658	10.8342	40
30	9.1697	9.9952	9.1745	10.8255	30
40	9.1781	9.9950	9.1831	10.8169	20
50	9.1863	9.9948	9.1915	10.8085	10
9° 00′	9.1943	9.9946	9.1997	10.8003	81° 00′
	L Cos	L Sin	L Cot	L Tan	Angle

* These tables give the logarithms increased by 10. Hence in each case 10 should be subtracted.

TABLE III. — LOGARITHMS OF TRIGONOMETRIC FUNCTIONS (*Cont.*)

Angle	L Sin	L Cos	L Tan	L Cot	
9° 00′	9.1943	9.9946	9.1997	10.8003	**81° 00′**
10	9.2022	9.9944	9.2078	10.7922	50
20	9.2100	9.9942	9.2158	10.7842	40
30	9.2176	9.9940	9.2236	10.7764	30
40	9.2251	9.9938	9.2313	10.7687	20
50	9.2324	9.9936	9.2389	10.7611	10
10° 00′	9.2397	9.9934	9.2463	10.7537	**80° 00′**
10	9.2468	9.9931	9.2536	10.7464	50
20	9.2538	9.9929	9.2609	10.7391	40
30	9.2606	9.9927	9.2680	10.7320	30
40	9.2674	9.9924	9.2750	10.7250	20
50	9.2740	9.9922	9.2819	10.7181	10
11° 00′	9.2806	9.9919	9.2887	10.7113	**79° 00′**
10	9.2870	9.9917	9.2953	10.7047	50
20	9.2934	9.9914	9.3020	10.6980	40
30	9.2997	9.9912	9.3085	10.6915	30
40	9.3058	9.9909	9.3149	10.6851	20
50	9.3119	9.9907	9.3212	10.6788	10
12° 00′	9.3179	9.9904	9.3275	10.6725	**78° 00′**
10	9.3238	9.9901	9.3336	10.6664	50
20	9.3296	9.9899	9.3397	10.6603	40
30	9.3353	9.9896	9.3458	10.6542	30
40	9.3410	9.9893	9.3517	10.6483	20
50	9.3466	9.9890	9.3576	10.6424	10
13° 00′	9.3521	9.9887	9.3634	10.6366	**77° 00′**
10	9.3575	9.9884	9.3691	10.6309	50
20	9.3629	9.9881	9.3748	10.6252	40
30	9.3682	9.9878	9.3804	10.6196	30
40	9.3734	9.9875	9.3859	10.6141	20
50	9.3786	9.9872	9.3914	10.6086	10
14° 00′	9.3837	9.9869	9.3968	10.6032	**76° 00′**
10	9.3887	9.9866	9.4021	10.5979	50
20	9.3937	9.9863	9.4074	10.5926	40
30	9.3986	9.9859	9.4127	10.5873	30
40	9.4035	9.9856	9.4178	10.5822	20
50	9.4083	9.9853	9.4230	10.5770	10
15° 00′	9.4130	9.9849	9.4281	10.5719	**75° 00′**
10	9.4177	9.9846	9.4331	10.5669	50
20	9.4223	9.9843	9.4381	10.5619	40
30	9.4269	9.9839	9.4430	10.5570	30
40	9.4314	9.9836	9.4479	10.5521	20
50	9.4359	9.9832	9.4527	10.5473	10
16° 00′	9.4403	9.9828	9.4575	10.5425	**74° 00′**
10	9.4447	9.9825	9.4622	10.5378	50
20	9.4491	9.9821	9.4669	10.5331	40
30	9.4533	9.9817	9.4716	10.5284	30
40	9.4576	9.9814	9.4762	10.5238	20
50	9.4618	9.9810	9.4808	10.5192	10
17° 00′	9.4659	9.9806	9.4853	10.5147	**73° 00′**
10	9.4700	9.9802	9.4898	10.5102	50
20	9.4741	9.9798	9.4943	10.5057	40
30	9.4781	9.9794	9.4987	10.5013	30
40	9.4821	9.9790	9.5031	10.4969	20
50	9.4861	9.9786	9.5075	10.4925	10
18° 00′	9.4900	9.9782	9.5118	10.4882	**72° 00′**
	L Cos	**L Sin**	**L Cot**	**L Tan**	**Angle**

Angle	L Sin	L Cos	L Tan	L Cot	
18° 00′	9.4900	9.9782	9.5118	10.4882	**72° 00′**
10	9.4939	9.9778	9.5161	10.4839	50
20	9.4977	9.9774	9.5203	10.4797	40
30	9.5015	9.9770	9.5245	10.4755	30
40	9.5052	9.9765	9.5287	10.4713	20
50	9.5090	9.9761	9.5329	10.4671	10
19° 00′	9.5126	9.9757	9.5370	10.4630	**71° 00′**
10	9.5163	9.9752	9.5411	10.4589	50
20	9.5199	9.9748	9.5451	10.4549	40
30	9.5235	9.9743	9.5491	10.4509	30
40	9.5270	9.9739	9.5531	10.4469	20
50	9.5306	9.9734	9.5571	10.4429	10
20° 00′	9.5341	9.9730	9.5611	10.4389	**70° 00′**
10	9.5375	9.9725	9.5650	10.4350	50
20	9.5409	9.9721	9.5689	10.4311	40
30	9.5443	9.9716	9.5727	10.4273	30
40	9.5477	9.9711	9.5766	10.4234	20
50	9.5510	9.9706	9.5804	10.4196	10
21° 00′	9.5543	9.9702	9.5842	10.4158	**69° 00′**
10	9.5576	9.9697	9.5879	10.4121	50
20	9.5609	9.9692	9.5917	10.4083	40
30	9.5641	9.9687	9.5954	10.4046	30
40	9.5673	9.9682	9.5991	10.4009	20
50	9.5704	9.9677	9.6028	10.3972	10
22° 00′	9.5736	9.9672	9.6064	10.3936	**68° 00′**
10	9.5767	9.9667	9.6100	10.3900	50
20	9.5798	9.9661	9.6136	10.3864	40
30	9.5828	9.9656	9.6172	10.3828	30
40	9.5859	9.9651	9.6208	10.3792	20
50	9.5889	9.9646	9.6243	10.3757	10
23° 00′	9.5919	9.9640	9.6279	10.3721	**67° 00′**
10	9.5948	9.9635	9.6314	10.3686	50
20	9.5978	9.9629	9.6348	10.3652	40
30	9.6007	9.9624	9.6383	10.3617	30
40	9.6036	9.9618	9.6417	10.3583	20
50	9.6065	9.9613	9.6452	10.3548	10
24° 00′	9.6093	9.9607	9.6486	10.3514	**66° 00′**
10	9.6121	9.9602	9.6520	10.3480	50
20	9.6149	9.9596	9.6553	10.3447	40
30	9.6177	9.9590	9.6587	10.3413	30
40	9.6205	9.9584	9.6620	10.3380	20
50	9.6232	9.9579	9.6654	10.3346	10
25° 00′	9.6259	9.9573	9.6687	10.3313	**65° 00′**
10	9.6286	9.9567	9.6720	10.3280	50
20	9.6313	9.9561	9.6752	10.3248	40
30	9.6340	9.9555	9.6785	10.3215	30
40	9.6366	9.9549	9.6817	10.3183	20
50	9.6392	9.9543	9.6850	10.3150	10
26° 00′	9.6418	9.9537	9.6882	10.3118	**64° 00′**
10	9.6444	9.9530	9.6914	10.3086	50
20	9.6470	9.9524	9.6946	10.3054	40
30	9.6495	9.9518	9.6977	10.3023	30
40	9.6521	9.9512	9.7009	10.2991	20
50	9.6546	9.9505	9.7040	10.2960	10
27° 00′	9.6570	9.9499	9.7072	10.2928	**63° 00′**
	L Cos	**L Sin**	**L Cot**	**L Tan**	**Angle**

Angle	L Sin	L Cos	L Tan	L Cot	
27° 00′	9.6570	9.9499	9.7072	10.2928	**63° 00′**
10	9.6595	9.9492	9.7103	10.2897	50
20	9.6620	9.9486	9.7134	10.2866	40
30	9.6644	9.9479	9.7165	10.2835	30
40	9.6668	9.9473	9.7196	10.2804	20
50	9.6692	9.9466	9.7226	10.2774	10
28° 00′	9.6716	9.9459	9.7257	10.2743	**62° 00′**
10	9.6740	9.9453	9.7287	10.2713	50
20	9.6763	9.9446	9.7317	10.2683	40
30	9.6787	9.9439	9.7348	10.2652	30
40	9.6810	9.9432	9.7378	10.2622	20
50	9.6833	9.9425	9.7408	10.2592	10
29° 00′	9.6856	9.9418	9.7438	10.2562	**61° 00′**
10	9.6878	9.9411	9.7467	10.2533	50
20	9.6901	9.9404	9.7497	10.2503	40
30	9.6923	9.9397	9.7526	10.2474	30
40	9.6946	9.9390	9.7556	10.2444	20
50	9.6968	9.9383	9.7585	10.2415	10
30° 00′	9.6990	9.9375	9.7614	10.2386	**60° 00′**
10	9.7012	9.9368	9.7644	10.2356	50
20	9.7033	9.9361	9.7673	10.2327	40
30	9.7055	9.9353	9.7701	10.2299	30
40	9.7076	9.9346	9.7730	10.2270	20
50	9.7097	9.9338	9.7759	10.2241	10
31° 00′	9.7118	9.9331	9.7788	10.2212	**59° 00′**
10	9.7139	9.9323	9.7816	10.2184	50
20	9.7160	9.9315	9.7845	10.2155	40
30	9.7181	9.9308	9.7873	10.2127	30
40	9.7201	9.9300	9.7902	10.2098	20
50	9.7222	9.9292	9.7930	10.2070	10
32° 00′	9.7242	9.9284	9.7958	10.2042	**58° 00′**
10	9.7262	9.9276	9.7986	10.2014	50
20	9.7282	9.9268	9.8014	10.1986	40
30	9.7302	9.9260	9.8042	10.1958	30
40	9.7322	9.9252	9.8070	10.1930	20
50	9.7342	9.9244	9.8097	10.1903	10
33° 00′	9.7361	9.9236	9.8125	10.1875	**57° 00′**
10	9.7380	9.9228	9.8153	10.1847	50
20	9.7400	9.9219	9.8180	10.1820	40
30	9.7419	9.9211	9.8208	10.1792	30
40	9.7438	9.9203	9.8235	10.1765	20
50	9.7457	9.9194	9.8263	10.1737	10
34° 00′	9.7476	9.9186	9.8290	10.1710	**56° 00′**
10	9.7494	9.9177	9.8317	10.1683	50
20	9.7513	9.9169	9.8344	10.1656	40
30	9.7531	9.9160	9.8371	10.1629	30
40	9.7550	9.9151	9.8398	10.1602	20
50	9.7568	9.9142	9.8425	10.1575	10
35° 00′	9.7586	9.9134	9.8452	10.1548	**55° 00′**
10	9.7604	9.9125	9.8479	10.1521	50
20	9.7622	9.9116	9.8506	10.1494	40
30	9.7640	9.9107	9.8533	10.1467	30
40	9.7657	9.9098	9.8559	10.1441	20
50	9.7675	9.9089	9.8586	10.1414	10
36° 00′	9.7692	9.9080	9.8613	10.1387	**54° 00′**
	L Cos	**L Sin**	**L Cot**	**L Tan**	**Angle**

Angle	L Sin	L Cos	L Tan	L Cot	
36° 00′	9.7692	9.9080	9.8613	10.1387	**54° 00′**
10	9.7710	9.9070	9.8639	10.1361	50
20	9.7727	9.9061	9.8666	10.1334	40
30	9.7744	9.9052	9.8692	10.1308	30
40	9.7761	9.9042	9.8718	10.1282	20
50	9.7778	9.9033	9.8745	10.1255	10
37° 00′	9.7795	9.9023	9.8771	10.1229	**53° 00′**
10	9.7811	9.9014	9.8797	10.1203	50
20	9.7828	9.9004	9.8824	10.1176	40
30	9.7844	9.8995	9.8850	10.1150	30
40	9.7861	9.8985	9.8876	10.1124	20
50	9.7877	9.8975	9.8902	10.1098	10
38° 00′	9.7893	9.8965	9.8928	10.1072	**52° 00′**
10	9.7910	9.8955	9.8954	10.1046	50
20	9.7926	9.8945	9.8980	10.1020	40
30	9.7941	9.8935	9.9006	10.0994	30
40	9.7957	9.8925	9.9032	10.0968	20
50	9.7973	9.8915	9.9058	10.0942	10
39° 00′	9.7989	9.8905	9.9084	10.0916	**51° 00′**
10	9.8004	9.8895	9.9110	10.0890	50
20	9.8020	9.8884	9.9135	10.0865	40
30	9.8035	9.8874	9.9161	10.0839	30
40	9.8050	9.8864	9.9187	10.0813	20
50	9.8066	9.8853	9.9212	10.0788	10
40° 00′	9.8081	9.8843	9.9238	10.0762	**50° 00′**
10	9.8096	9.8832	9.9264	10.0736	50
20	9.8111	9.8821	9.9289	10.0711	40
30	9.8125	9.8810	9.9315	10.0685	30
40	9.8140	9.8800	9.9341	10.0659	20
50	9.8155	9.8789	9.9366	10.0634	10
41° 00′	9.8169	9.8778	9.9392	10.0608	**49° 00′**
10	9.8184	9.8767	9.9417	10.0583	50
20	9.8198	9.8756	9.9443	10.0557	40
30	9.8213	9.8745	9.9468	10.0532	30
40	9.8227	9.8733	9.9494	10.0506	20
50	9.8241	9.8722	9.9519	10.0481	10
42° 00′	9.8255	9.8711	9.9544	10.0456	**48° 00′**
10	9.8269	9.8699	9.9570	10.0430	50
20	9.8283	9.8688	9.9595	10.0405	40
30	9.8297	9.8676	9.9621	10.0379	30
40	9.8311	9.8665	9.9646	10.0354	20
50	9.8324	9.8653	9.9671	10.0329	10
43° 00′	9.8338	9.8641	9.9697	10.0303	**47° 00′**
10	9.8351	9.8629	9.9722	10.0278	50
20	9.8365	9.8618	9.9747	10.0253	40
30	9.8378	9.8606	9.9772	10.0228	30
40	9.8391	9.8594	9.9798	10.0202	20
50	9.8405	9.8582	9.9823	10.0177	10
44° 00′	9.8418	9.8569	9.9848	10.0152	**46° 00′**
10	9.8431	9.8557	9.9874	10.0126	50
20	9.8444	9.8545	9.9899	10.0101	40
30	9.8457	9.8532	9.9924	10.0076	30
40	9.8469	9.8520	9.9949	10.0051	20
50	9.8482	9.8507	9.9975	10.0025	10
45° 00′	9.8495	9.8495	10.0000	10.0000	**45° 00′**
	L Cos	L Sin	L Cot	L Tan	Angle

HELPFUL MATERIAL FOR TEACHERS

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Teaching of Algebra, The (Seventh yearbook of the National Council of Teachers of Mathematics) Bureau of Publications, Teachers College, Columbia University, 1932.

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Breslich, R. F. *The Teaching of Mathematics in Secondary Schools*, University of Chicago Press, 1930. *Problems in Teaching Secondary School Mathematics*, University of Chicago Press, 1931. *Administration of Mathematics in Secondary Schools*, University of Chicago Press, 1933.

Hassler, J. O. and Smith, R. R. *The Teaching of Mathematics*, Macmillan Co., New York City, 1930.

Young, J. W. *Lectures on Fundamental Concepts of Algebra and Geometry*, Macmillan Co., 1911.

Westaway, F. W. *Craftsmanship in the Teaching of Elementary Mathematics*, Blackie and Son, Ltd., 1931.

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TYPICAL EXAMINATIONS

The examinations given in this section serve to illustrate the type of examination required of students who have taken a fusion course in Intermediate Algebra and Plane Trigonometry.

EXAMINATION I

Part I

1. Find the value of $(3x)^0 + x^{-\frac{2}{3}}$, when $x = 8$.

2. Express with a rational denominator the fraction $\dfrac{5}{3 - \sqrt{3}}$.

3. Find the sum of the fractions $\dfrac{b}{b^2 - a^2}$ and $\dfrac{1}{a - b}$.

4. Write the first three terms of the expansion of $(x - a)^8$.

5. What is the positive geometric mean between 4 and 25?

6. When two cells are connected in series, the electric current in the circuit can be computed from the formula $i = \dfrac{2E}{R + 2r}$, in which E represents the voltage, r the internal resistance, and R the external resistance. Solve this formula for r.

7. The legs of a right triangle are a and b and the hypotenuse is c. Express c as a function of a and b; that is, express c in terms of a and b.

8. Given the equation $x^2 - 6x + 9 = 0$; if the constant term 9 is increased, while the coefficients of the other terms remain the same, the roots of the resulting equation become (a) real and equal, (b) real and unequal, or (c) imaginary. Which is correct, (a) or (b) or (c)?

9. Write the equation of the straight line whose slope is $\frac{1}{2}$ and whose y-intercept is 1.

10. Find the radius of the circle whose center is at the point (0, 0) and which passes through the point (1, 2).

11. Express sin 280° as a function of a positive acute angle.

12. If $\cos A$ is negative and $\sin A = -\frac{1}{2}$, what is the value of $\cot A$?

13. Find to the nearest degree angle A of triangle ABC if $a = 2$, $b = 3$, and $c = 4$.

14. Find side a of triangle ABC if $b = 5$, $B = 30°$, and $A = 43°$.

15. Express 135° in radian measure. [Answer may be left in terms of π.]

16. Express the area K of triangle ABC as a function of a, b, and C.

17. What value of A between 90° and 180° satisfies the equation $2\sin^2 A + 7\sin A - 4 = 0$?

18. Is the following statement true or false? If in the triangle ABC, $A = 30°$, $b = 14$, and $a = 15$, the triangle is determined as to size and shape.

19. Is the tangent of a positive acute angle ever less than the sine of that angle? [Answer *Yes* or *No*.]

20. When plotted on the same pair of axes for values of x from 0° to 360°, in how many points does the graph of $y = \sin x$ cross the graph of $y = \cos x$?

Part II

Group I

Answer three questions from this group.

21. A cross-country team ran 6 miles at a constant rate and then returned at a rate 5 miles less per hour. At what rates did they run if they were 50 minutes longer in returning than in going?

22. A nurse prepared 10 ounces of a disinfectant that was 15% carbolic acid. How much water must she add to reduce it to a 6% solution?

23. A principal of \$150, deposited in a trust company, bears interest at 4%, compounded semiannually. Using the formula $A = P\left(1 + \frac{r}{2}\right)^{2n}$, compute to the nearest dollar the amount A, which had accumulated at the end of 12 years.

24. Given the equation $x^2 + kx + k = 0$:

(a) Express the discriminant d of the given equation as a function of k; that is, express d in terms of k.

(b) Using the vertical axis to represent d and the horizontal axis to represent k, plot the graph of the function obtained in answer to (a), using values of k from -1 to $+5$ inclusive.

(c) On the graph made in answer to (b), indicate the points at which the values of k will produce a zero discriminant.

25. Solve the following set of equations for x, y, and z:

$$\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$$
$$\frac{1}{y} + \frac{1}{z} = \frac{8}{15}$$
$$\frac{1}{x} + \frac{1}{z} = \frac{7}{10}.$$

Group II

Answer two questions from this group.

26. To find the distance BC across a swamp, surveyors located a point A which was 80.7 feet from B and 110.3 feet from C. The angle BAC was measured and found to be $52°\ 0'$. Find the distance across the swamp.

27. In the triangle ABC, $A = 54°\ 15'$, $B = 68°\ 20'$, and $AB = 10$. Find the altitude on side AB.

28. (a) If $x = \sin^{-1} \frac{3}{5}$ and $y = \cos^{-1} \frac{5}{13}$, both angles being acute, find by the use of the proper formula the value of $\sin(x + y)$.

(*b*) Prove the following identity: $(\sin x + \cos x)^2 = 1 + \sin 2x$.

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EXAMINATION II

Part I

1. Write in simplest form the value of $4^{\frac{3}{2}} \times 2y^0$.

2. Write the sixth term of the expansion of $(3x - y)^7$.

3. What term of the progression $7, 4, 1, \cdots$ is -65?

4. Solve for x the following equation: $2x = \frac{1}{x} + \frac{7}{2}$.

5. Solve for x the following equation: $\sqrt{4x^2 + 5} + 2x = 1$.

6. Write in the form $ax^2 + bx + c = 0$ the equation whose roots are $1 + \sqrt{3}$ and $1 - \sqrt{3}$.

7. Make a formula for the cost (C) of riding m miles in a taxicab at the rate of 25 cents for the first $2\frac{1}{2}$ miles and f cents for each succeeding half mile.

8. Solve for x, correct to the nearest tenth: $10^x = 28$.

9. Write in exponential form: $\log_{10} R = K$.

10. Simplify $\dfrac{a - b}{\frac{a}{b} - \frac{b}{a}}$.

11. Find $\cot 37° 28'$.

12. Find to the nearest minute the smallest positive value of A if $\log \sin A = 9.8621 - 10$.

13. If $\cos A = -\frac{5}{13}$ and A is in the second quadrant, find $\tan A$.

14. Solve for a positive value of x in the third quadrant: $\sin^2 x = \frac{1}{2}$.

15. If A is in the second quadrant and $\sin A$ increases, does A increase or decrease?

16. Express in radical form the value of $\cot 300°$.

17. The length of a rectangle is 200 feet. If one of the angles

between the diagonals is 140°, find to the nearest foot the width of the rectangle.

18. Express in terms of a single trigonometric function of $2A$:

$$\frac{\cos^2 A - \sin^2 A}{\sin A \cos A}.$$

19. Find to the nearest degree the acute angle made with the x-axis by the graph of the equation $3x - 4y = 12$.

20. What is the largest positive value that $\cos 5A$ may have?

Part II

Group I

Answer three questions from this group.

21. An auditorium balcony is to seat 490 persons. The first row has 40 seats and each succeeding row has two more seats than the row in front of it. How many rows must there be?

22. The hypotenuse of a right triangle is 26 and the sum of the other two sides is 34. Find the length of the sides.

23. A man deposits $1000 in a savings bank on the day when his son is born. With interest at 4% compounded semiannually, how much should the son receive at the end of his twenty-first year?

24. A dealer bought some melons for $2.04. After throwing away 4 that were bad, he sold the rest at 6 cents apiece more than he paid for them and made a total profit of 22 cents. How many melons did he buy?

25. Determine graphically whether the following equations have any common solutions:

$$y = 2^x$$
$$2y = x.$$

Group II

Answer two questions from this group.

26. The distance from A to a point C due west of A is known to be less than 500 feet. Previous measurements from a point B

have given $BA = 675.8$ feet, $BC = 610.3$ feet; the bearing of B from A is N 48° 37′ W. Find the distance AC correct to the nearest tenth of a foot.

27. Determine to the nearest degree the positive value of A less than 180° that satisfies the equation $4 \sin A - 7 \cos A = 1$.

28. A tower 100 feet high stands on the seashore. From its top the angle of depression of a boat is 18° 40′. The distance from the foot of the tower to an island is 704 feet. Find the distance from the boat to the island if this distance subtends at the foot of the tower a horizontal angle of 107°.

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EXAMINATION III

Part I

1. In which quadrant is the angle $\frac{7\pi}{5}$ radians?

2. As an angle increases from 0° to 90°, which three of its six trigonometric functions increase?

3. Find the arithmetic mean between $3 \cos x$ and $-\cos x$.

4. Find the logarithm of 14.87.

5. Express cos 335° as a function of a positive angle less than 90°.

6. Write the quadratic equation with integral coefficients whose roots are $\frac{3}{2}$ and -1.

7. Find the tangent of the angle which the graph of $y = 3x - 7$ forms with the x-axis.

8. Solve the equation $x^2 - 2x + 2 = 0$ and write the answers in terms of i.

9. Solve the equation $\dfrac{3}{\sqrt{\tan x + 4}} = \sqrt{\tan x + 4}$ for the value of x in the second quadrant.

10. In the triangle ABC, if $a = 7$, $b = 5$, $c = 6$, find the value of $\cos A$.

11. Find the value of $27^{-\frac{2}{3}} + 64^0$.

12. For what values of angles A between $0°$ and $360°$ does the graph of $\cos A$ cross the x-axis?

13. If $\tan A = a$ and $\tan B = -\frac{1}{a}$, find the value of $\tan (A + B)$.

14. If x varies as y^2, and $x = 4$ when $y = 8$, find x when $y = 2$.

15. Write the first three terms of the expansion of $(c - d)^5$.

16. Given $A = \sin^{-1}\frac{1}{3}$, A being a positive obtuse angle; find A correct to the nearest minute.

17. Write the equation of the straight line that passes through the point (2, 9) and has the slope 2.

18. Is it possible to have a triangle ABC in which $a = 21$, $b = 32$, $A = 115°$? [Answer *Yes* or *No*.]

19. In the series $\sin x$, $\tan x$, $\sin x \sec^2 x$, $\cdots$ find the ratio in terms of $\cos x$.

20. If $\log \cos A = 9.8119 - 10$, find angle A correct to the nearest minute.

Part II

Group I

Answer three questions from this group.

21. Solve the following set of equations and group your answers:

$$x^2 + xy + y^2 = 39$$
$$x + y + 2 = 0.$$

22. Two trains run at uniform rates for a distance of 120 miles. One train travels 5 miles an hour faster than the other and takes 20 minutes less time to travel the distance. Find the rate of the faster train.

23. A machine is worth at the end of each year only 90% of what it was worth at the beginning of the same year. If the machine cost $1500, find to the nearest dollar its value at the end of 6 years. [Solve by using the proper progression formula.]

24. How many pounds of a 4% solution of salt must be added to 25 pounds of a 12% solution in order to obtain a mixture that is 10% salt?

25. (*a*) Plot the graph of $x^2 + y^2 = 25$.

(*b*) On the same set of axes used in answer to (*a*), plot the graph of $xy = 12$ from $x = -6$ to $x = +6$ inclusive.

(*c*) From the graphs made in answer to (*a*) and (*b*), obtain the values of x and y that satisfy both equations.

Group II

Answer two questions from this group.

26. (*a*) Prove the identity: $\sec A + \csc A = \dfrac{1 + \tan A}{\sin A}$.

(*b*) Solve the following equation for values of x between 90° and 270°: $\cos 2x \sec x + \sec x + 1 = 0$.

27. Two straight railroad tracks that form an angle of 74° with each other start from the same station. Two trains leave this station at the same time, one on each track, and run at rates of 35 and 45 miles an hour. Find to the nearest mile the distance between the two trains at the end of 30 minutes.

28. A flagstaff s feet high stands on the top of a tower. From a point in the plane on which the tower stands, the angles of elevation of the top and bottom of the flagstaff are observed to be A and B respectively. Show that the height h of the tower is given by the formula $h = \dfrac{s \tan B}{\tan A - \tan B}$.

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